Stochastic resonance: non-robust and robust tuning notions*

S. Herrmann[1] P. Imkeller[2] I. Pavlyukevich[3]

14th February 2003

Abstract

We consider a dynamical system describing the diffusive motion of a particle in a double well potential with a periodic perturbation of very small frequency, and an additive stochastic perturbation of amplitude ε. It is in stochastic resonance if the solution trajectories amplify the small periodic perturbation in a 'best possible way'. Systems of this type first appeared in simple energy balance models designed for a qualitative explanation of global glacial cycles. Large deviations theory provides a lower bound for the proportion of the amplitude and the logarithm of the period above which quasi-deterministic periodic behavior can be observed. To obtain optimality, one has to measure periodicity with a measure of quality of tuning. Notions of quality of tuning widely used in physics such as the *spectral power amplification* or the *signal-to-noise ratio* depend on the spectral properties of the averaged trajectories of the diffusion. We show that these notions pose serious mathematical problems if the underlying system is reduced to simpler Markov chain models on the finite state space composed of the meta-stable states of the potential landscape in the limit of small noise. As a way out of this dilemma we propose to measure the quality of periodic tuning by the probability that transitions between the domains of attraction of the potential wells happen during a parametrized time window maximized in the window parameter. This notion can be investigated by means of uniform large deviations estimates and turns out to be robust for the passage to dimension reduced Markov chains.

AMS 2000 Classification: 60H10, 60J60, 60J27, 86A04, 35P15, 37A30, 37N10
Keywords: Stochastic resonance, spectral gap, stochastic differential equation, energy balance model, potential diffusion, effective dynamics, large deviations, exit time, domain of attraction.

[1]Université Henri Poincaré Nancy I, B.P. 239, 54506 Vandoeuvre-lès-Nancy Cedex, France, `email:herrmann@iecn.u-nancy.fr`

[2]Institut für Mathematik, Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany, `email:imkeller@mathematik.hu-berlin.de`

[3]Institut für Mathematik, Technische Universität Berlin, Straße des 17. Juni, 136, 10623 Berlin, Germany, `email:pavljuke@math.tu-berlin.de`

*Supported by the DFG research center 'Mathematics for key technologies' (FZT 86) in Berlin.

1 Background and paradigm

Usually speaking about *noise* we mean something that deteriorates the operation of a system. In common language noise is understood as a disturbance, especially a random and persistent disturbance, that obscures or reduces the clarity of a signal.

However, in nonlinear systems, in particular climate systems, the presence of noise may play a very constructive role. This paper is devoted to a survey of mathematical approaches to study systems displaying *stochastic resonance* (SR), in which an essentially non-zero level of noise enhances the systems' sensitivity and ability to recover and amplify small periodic deterministic signals. In fact, in the example of the simple model of the dynamics of *glacial cycles* in paleoclimate of the earth we shall discuss in more detail below, only the appearance of noise forcing is able to reproduce qualitatively important features of transitions between different basic climate states such as ice and warm ages.

It was no surprise that the phenomenon of stochastic resonance appearing in numerous examples from electronics to biology (see [13]) was recently rediscovered in higher dimensional climate systems by Ganopolski and Rahmstorf [14]. In the analysis of the Greenland ice core record, the statistical properties of spontaneous intermediate warmings which are commonly known as Dansgaard-Oeschger events, were found to be consistent with stochastic resonance. It is observed that besides the meta-stable ice and warm age temperature states with transition times around multiples of $10^4 - 10^5$ years there is another meta-stable state at an intermediate temperature accessible from the glacial state. Transition intervals cluster around integer multiples of 1500 years. Ganopolski and Rahmstorf reproduce these observations by a simulation based on the

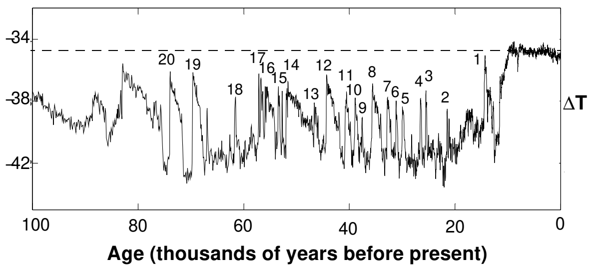

Fig. 1: Record of the oxygen isotope concentration from the GRIP ice core. Dansgaard-Oeschger warm events are numbered [14].

CLIMBER coupled ocean-atmosphere model of moderate complexity established by the Potsdam group. A stability analysis shows the existence of the intermediate metastable state, and suitable small periodic and random excitations of the salinity balance of the North Atlantic produce temperature curves with abrupt transitions of the observed type. The empirical distribution of the interspike intervals (i.e. multiples of the basic transition period) is seen to be a function of noise amplitude. Recent work by Paillard [23] indicates that global temperature fluctuations beyond 10^6 years back in earth's history may be interpreted as noise induced transitions in tri-stable situations.

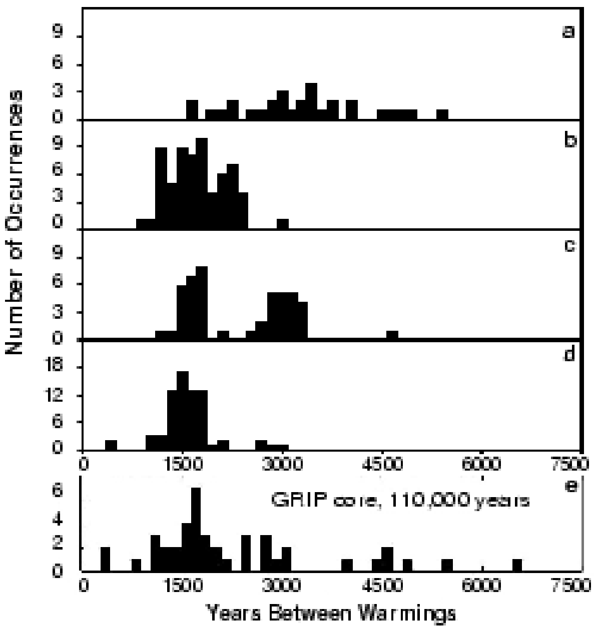

Fig. 2: Interspike interval distribution (or waiting time between warm events) for 'noise only' and 'noise plus signal' numerical experiments. The bottom chart represents a real Greenland ice core record [14].

The paradigm of *stochastic resonance* emerged from papers by C. Nicolis [21] and Benzi et al. [2, 3, 4]. The model they designed is based on the following observations. Modern measurement techniques allow to determine ratios of oxygen isotope concentrations O^{18}/O^{16} in deep sea core sediments which in turn provide rough estimates of the *global mean temperature* of the earth at the time they were deposited. This way at least seven changes between 'cold' and 'warm' periods were detected during approximately the last 700,000 years. They occur spontaneously and abruptly with surprisingly short relaxation times and with roughly the same period of about 10^5 years. The quoted papers aimed at suggesting a simple mathematical model to account for this deterministic-looking periodicity on the one hand, and for the intriguing spontaneous transition mechanism on the other hand.

The proposed model just appeals to conservation of radiative energy and supposes that the earth's global temperature T satisfies a simple energy-balance equation (for details see [16, 17]), i.e. the instant change of the global temperature is proportional to the difference between *incoming* and *outgoing* radiative energy:

$$c\frac{dT(t)}{dt} = Q(t)(1 - a(T(t))) - \sigma T(t)^4, \quad c > 0. \tag{1.1}$$

In the simplest case considered here it is assumed that the total energy flux emitted by the earth is given by the Stefan–Boltzmann law valid for a black body radiator.

The absorbed energy depends on two factors. The *global solar* function $Q(t)$ describes the flux of the solar energy which reaches the earth at time t. Assuming that the solar activity is a constant Q_0, the function Q depends on the distance between the earth and the sun as well as on the inclination of the earth's axis, and due to the gravitational influence of Jupiter exhibits a slow periodic variation of a period of about 10^5 years. The variation is estimated to be 0.1% of Q_0. Thus we put

$$Q(t) = Q_0 - b \sin \omega t, \quad b \approx 0.001 Q_0, \quad \omega \approx (2\pi 10^5)^{-1} [\mathrm{yr}^{-1}].$$

On the other hand, not all the solar radiation reaching the atmosphere is absorbed: the proportion of absorbed radiation is determined by the earth's albedo a which depends *locally* on the earth's average surface temperature T. The simple albedo model employed here appears in the papers by Budyko [7] and Sellers [25], see Fig. 3 (l.). $a(T)$ is chosen to be a constant close to 1 for low temperatures $T \leq \underline{T}$. In this

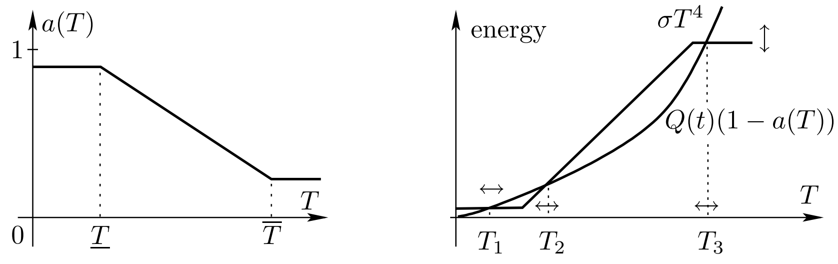

Fig. 3: Earth's albedo (l.) and incoming and outgoing radiative energies (r.).

temperature regime all surface water is supposed to be frozen and to cover a big portion of the planet by a bright ice layer making the reflection ratio relatively high. For high temperatures $T \geq \overline{T}$ the green-brown vegetation results in a low reflection ratio. In the regime between $\underline{T}$ and $\overline{T}$ the function $a(T)$ is interpolated linearly.

Thus, the right hand side of (1.1) is a difference of two functions, see Fig. 3 (r.). For appropriate values of parameters the dynamical system (1.1) has two meta-stable equilibrium states T_1 and T_3 separated by the unstable state T_2. The lower metastable state T_1 is interpreted as describing ice age temperatures whereas T_3 determines warm ages.

This model of climate has major shortcomings and therefore cannot picture reality. Indeed, solutions of (1.1) converge to either T_1 or T_3 and oscillate with periods of 10^5 years with relatively small amplitudes, due to the smallness of b. Most importantly, however, the typically observed spontaneous and rapid transitions between 'cold' and 'warm' states are impossible.

To overcome this difficulty C. Nicolis and Benzi et al. added a *noise* term to the energy-balance equation (1.1) and obtained the following simple SDE for the global temperature:

$$c \frac{dT(t)}{dt} = Q(t)(1 - a(T(t))) - \sigma T(t)^4 + \sqrt{\varepsilon} \dot{W}_t, \quad c > 0, \tag{1.2}$$

where W is a standard one-dimensional Brownian motion and $\varepsilon > 0$. In this setting, transitions between the meta-stable climate states become possible, and — most importantly — relaxation times are small (of the order 10^2 years) and much more realistic.

In the one-dimensional setting of the system (1.2) one can always represent the drift term as a gradient in the variable T of some potential function U, i.e. we can find U such that

$$-\frac{\partial U(T,t)}{\partial T} = Q(t)(1 - a(T(t))) - \sigma T(t)^4.$$

Of course, the potential depends on time. Fig. 4 shows the incoming and outgoing radiation, their difference, and the corresponding potential function at times when the solar constant takes its minimum (left column) and maximum (right column). In

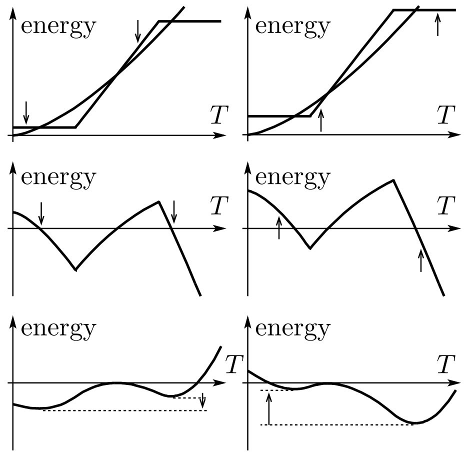

Fig. 4: The drifts of (1.2) and the corresponding potentials at times when the solar constant takes its minimum (l.) and maximum (r.).

terms of the potential U, equation (1.2) describes the dynamics of an overdamped Brownian particle in a double well potential, where the minima of the potential wells correspond to the 'cold' and 'warm' global temperatures. The depths of the potential wells vary periodically in time, and the left well is deeper for approximately 5×10^4 years. Clearly, this periodic and deterministic variation of the wells' depths is the most important feature of the potential U.

2 Periodically hopping potentials and the defect of spectral resonance notions

To catch the essentials of the effect and at the same time to simplify the problem we will work in the first part of this paper with a time-space asymmetric double well potential switching discontinuously between two states. In the second part we will essentially extend this framework to include continuously varying potentials. In the strip $(x,t) \in \mathbb{R} \times [0,1)$ it is defined by the formula

$$U(x,t) = \begin{cases} U_1(x), & t \in [0, \frac{1}{2}), \\ U_2(x) = U_1(-x), & t \in [\frac{1}{2}, 1). \end{cases} \tag{2.1}$$

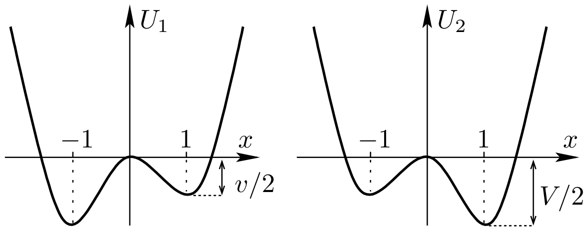

Fig. 5: Time-periodic potential U.

It is periodically extended for all times t by the relation $U(\cdot, t) = U(\cdot, t+1)$, see Fig. 5. We assume that the potential has two local minima at ± 1 and a local maximum at 0, that $U_1(-1) = -\frac{V}{2}$, $U_1(1) = -\frac{v}{2}$, $\frac{2}{3} < \frac{v}{V} < 1$, and $U_1(0) = 0$. We also suppose that the extrema of U are not degenerate, i.e. the curvatures at these points do not vanish.

A trajectory of a Brownian particle in this potential is described by the SDE

$$dX_t^{\varepsilon,T} = -U'(X_t^{\varepsilon,T}, \frac{t}{T})\, dt + \sqrt{\varepsilon}\, dW_t, \quad X_0^{\varepsilon,T} = x \in \mathbb{R}, \tag{2.2}$$

where $\varepsilon > 0$ is the noise intensity, and $T > 0$ the period.

The problem of finding an intensity characterizing stochastic resonance now consists in determining an *optimal tuning* $\varepsilon = \varepsilon(T)$, i.e. the noise intensity for which the trajectories $X^{\varepsilon,T}$ look 'as periodic as possible'. Of course, in these terms stochastic resonance is a rather unprecise concept. To make it precise at least requires *measuring* periodicity in diffusion trajectories.

2.1 Freidlin's quasi-deterministic periodic response

Using large deviations theory, M. Freidlin [11] explains periodicity as a quasi-deterministic property of diffusion trajectories for very large period lengths.

Consider, for example, a Brownian particle in the time homogeneous double well potential U_1 described by the SDE

$$dX_t^{\varepsilon} = -U_1'(X_t^{\varepsilon})\, dt + \sqrt{\varepsilon}\, dW_t. \tag{2.3}$$

For small ε, this stochastic system can be considered as a small white noise perturbation of the deterministic dynamical system $\dot{x} = -U_1'(x)$. The Freidlin-Wentzell theory of large deviations [12] allows to study asymptotic properties of (2.3) as $\varepsilon \to 0$ in terms of the geometric properties of the potential U_1. It is intuitively clear that for small noise intensities the sample paths of (2.3) spend most of the time in small neighborhoods of the *meta-stable* states ± 1. Jumps between the wells occur, but very rarely. The probability of these transitions can be estimated in terms of the so-called *quasipotential* which measures the work to be done by the diffusion in order to travel between different points in the potential landscape. Let, for instance, $X_0^{\varepsilon} = -1$ and x belong to the left well. Then the quasipotential $V(-1, x)$ can be found explicitly and equals $2(U_1(x) - U_1(-1))$ and thus twice the height of the potential barrier between -1 and x. If x belongs to the right well and $0 < x \leq 1$, then $V(-1, x) = 2(U_1(0) - U_1(-1))$. Only the way 'up' in the potential landscape contributes to the quasipotential; the way 'down' requires no work and is free. Quasipotentials are defined for rather general classes of stochastic systems by means of *action functionals*, for details see [12].

Let us define the first entrance time

$$\tau_y(X^\varepsilon) = \inf\{t \geq 0 \, : \, X_t^\varepsilon = y\}.$$

Then the quasipotential at x and y determines the exponential order of $\tau_y(X^\varepsilon)$ if the diffusion starts in x (under the law $\mathbf{P}_x$) in the limit of small noise, see [12, 11].

Theorem 2.1 ('transition law') *For all $\delta > 0$ the following holds:*

$$\lim_{\varepsilon \downarrow 0} \mathbf{P}_x\left(e^{\frac{1}{\varepsilon}(V(x,y)-\delta)} < \tau_y(X^\varepsilon) < e^{\frac{1}{\varepsilon}(V(x,y)+\delta)}\right) = 1.$$

The most important statement of the theorem is that the system (2.3) has two exponentially different intrinsic time scales: the exit time from the left well is of the order $e^{V/\varepsilon}$ whereas the exit time from the right well of the order $e^{v/\varepsilon}$. This results in the following observation: if we consider the trajectories of (2.3) on the exponentially long time intervals $T_\varepsilon \propto e^{\omega/\varepsilon}$, then for $0 < \omega < v$ the trajectory typically does not leave its initial well, and for $\omega > v$ it spends most of its time near -1 (in probability). In other words, on the different time scales the system (2.3) has different meta-stable states.

This description of meta-stable behaviour can be transferred to the time inhomogeneous system (2.2). Let the period $T = T_\varepsilon$ be such that $\lim_{\varepsilon \downarrow 0} \varepsilon \ln T_\varepsilon = \omega > 0$. Then for $\omega < v$ the diffusion does not have enough time to leave even the shallow well during one half period, and therefore, as in the time homogeneous case, does not leave its initial well. However, if $\omega > v$, a new effect appears.

Theorem 2.2 (Freidlin, [11]) *Let the process $X^{\varepsilon,T}$ satisfy (2.2), and*

$$\lim_{\varepsilon \downarrow 0} \varepsilon \ln T_\varepsilon > v. \tag{2.4}$$

Then for all $A > 0$ and $\delta > 0$ the following holds true:

$$\Lambda\{t \in [0, A] \, : \, |X_{T_\varepsilon t}^{\varepsilon,T} - \phi(t)| > \delta\} \to 0$$

in $\mathbf{P}_x$-probability as $\varepsilon \to 0$, where $\Lambda\{\cdot\}$ denotes Lebesgue measure on $\mathbb{R}$, and

$$\phi(t) = \begin{cases} -1, & t \pmod 1 \in [0, \frac{1}{2}), \\ 1, & t \pmod 1 \in [\frac{1}{2}, 1), \end{cases} \tag{2.5}$$

is the coordinate of the global minimum of $U(\cdot, t)$, see Fig. 6.

Theorem 2.2 suggests one possible measure of periodicity of diffusion trajectories: take the Lebesgue measure of those times the trajectories spend outside of a δ-tube around the deterministic discontinuous periodic function ϕ. Condition (2.4) on period T_ε and noise intensity ε provides a family of tunings, without, however, suggesting an optimal one to determine the resonance point. This is illustrated by Fig. 6 (r.) which clearly suggests that for large T_ε excursions to the 'wrong' well are not very long but frequent and destroy a periodic picture.

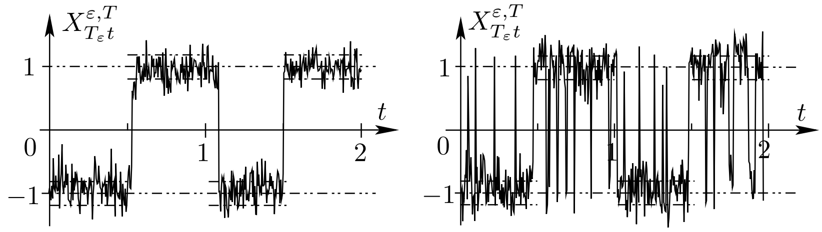

Fig. 6: On time intervals satisfying condition (2.4) the diffusion $X^{\varepsilon,T}$ is close to the deterministic periodic function ϕ.

2.2 Spectral power amplification

The coefficient of *spectral power amplification* (SPA) is one of the physicists' favorite measures to measure periodicity of random trajectories, see e.g. [4, 20, 8, 13, 1, 26]. It is based on the power spectrum of the average trajectories of the diffusion (2.2) with respect to equilibrium and, even in the simple setting chosen, has serious defects when compared to an analogous notion for the dimension reduced Markov chain describing the *effective dynamics*, i.e. its *interwell* dynamics, the hopping between the two metastable states, while neglecting the *intrawell* small fluctuations. The SPA coefficient is defined by

$$\eta^X(\varepsilon, T) = \left| \int_0^1 \mathbf{E}_\mu(X^{\varepsilon,T}_{Ts}) \cdot e^{2\pi i s}\, ds \right|^2. \tag{2.6}$$

The function η^X having noise intensity and the period of time variation of the potential as arguments has a clear physical meaning. It shows how much energy is carried by the averaged path of the diffusion with noise amplitude ε on the frequency $\frac{2\pi}{T}$. The expectation $\mathbf{E}_\mu$ indicates that averages are taken with respect to the time-periodic equilibrium measure of $X^{\varepsilon,T}$. This will be explained in detail later.

Fig. 7 borrowed from [1] where Ω corresponds to our $\frac{2\pi}{T}$ and D to the diffusion intensity ε shows that physicists expect a local maximum of the function $\varepsilon \mapsto \eta^X(\varepsilon, \cdot)$. The random paths have their strongest periodic component at the value of ε for which the maximum is taken. In fact, Fig. 7 depicts not the SPA coefficient of the diffusion itself, but of its so-called 'effective dynamics'. It is *a priori* believed in the physical literature that the effective dynamics adequately describes the properties of the diffusion in the limit of small noise.

To determine the 'optimal tuning' or *stochastic resonance point* if periodic tuning is measured by SPA means to find the argument $\varepsilon = \varepsilon(T)$ of a local maximum of $\varepsilon \mapsto \eta^X(\varepsilon, \cdot)$.

The key to the solution of this problem lies in determining the time-dependent invariant density μ of $(X^{\varepsilon,T}_{Tt})_{t\geq 0}$. From now on we follow [24] and [19]. Although the diffusion is not time homogeneous, by enlarging its state space we can consider a two-dimensional time homogeneous Markov process $(X^{\varepsilon,T}_{Tt}, t \pmod 1)$ which possesses an invariant law in the usual sense. By definition we identify the time-dependent equilibrium density μ of $(X^{\varepsilon,T}_{Tt})_{t\geq 0}$ with the invariant density of the two-dimensional process. Indeed, with respect to μ and for fixed t, the law of the real random variable

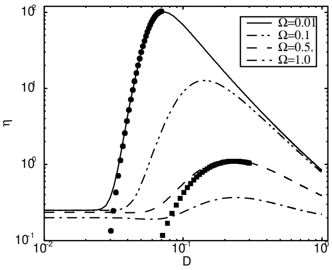

Fig. 7: SPA coefficient as a function of noise amplitude has a well pronounced maximum depending on the frequency of periodic perturbation [1].

$X^{\varepsilon,T}_{Tt}$ has the density $\mu(\cdot, t \pmod 1)$. The invariant density μ is a positive solution of the forward Kolmogorov (Fokker–Planck) equation $A^*_{\varepsilon,T}\mu = 0$, where

$$A^*_{\varepsilon,T}\cdot = -\frac{1}{T}\frac{\partial}{\partial t}\cdot + \frac{\varepsilon}{2}\frac{\partial^2}{\partial x^2}\cdot + \frac{\partial}{\partial x}\left(\cdot \frac{\partial}{\partial x}U\right)$$

is the formal adjoint of the infinitesimal generator of the two-dimensional diffusion. Moreover, from the time periodicity and time-space antisymmetry of the potential U (2.1) one concludes that $\mu(x,t) = \mu(-x, t+\frac{1}{2})$ and $\mu(x,t) = \mu(x, t+1)$, $(x,t) \in \mathbb{R}\times\mathbb{R}_+$.

This results in the following boundary-value problem used to determine μ. It is enough to solve the Fokker–Planck equation $A^*_{\varepsilon,T}\mu = 0$ in the strip $(x,t) \in \mathbb{R}\times[0, \frac{1}{2}]$ with boundary condition $\mu(x,0) = \mu(-x, \frac{1}{2})$, $x \in \mathbb{R}$.

2.3 The spectral gap

We have assumed in (2.1) that the time dependent potential U is a step function of the time variable. In the region $(x,t) \in \mathbb{R}\times(0, \frac{1}{2})$ it is identical to a time independent double well potential U_1, and therefore the Fokker–Planck equation turns into a one-dimensional parabolic PDE

$$\frac{1}{T}\frac{\partial}{\partial t}\mu(x,t) = \frac{\varepsilon}{2}\frac{\partial^2}{\partial x^2}\mu(x,t) + \frac{\partial}{\partial x}\left(\mu(x,t)\frac{\partial}{\partial x}U_1(x)\right). \tag{2.7}$$

Let L^*_ε denote the second order differential operator appearing on the right hand side of (2.7).

To determine μ we shall use the Fourier method of separation of variables which consists in expanding the solution of (2.7) into a Fourier series with respect to the system of eigenfunctions of the operator L^*_ε. It turns out that under the condition that U_1 is smooth and increases at least super-linearly at $\pm\infty$, the operator L^*_ε is essentially self-adjoint in $\mathcal{L}^2(\mathbb{R}, e^{\frac{2U_1}{\varepsilon}}\,dx)$, its spectrum is discrete and non-positive, and the corresponding eigenspaces are one-dimensional. Denoting by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ the norm and the inner product in $\mathcal{L}^2(\mathbb{R}, e^{\frac{2U_1}{\varepsilon}}\,dx)$ we consider the following formal Floquet type expansion

$$\mu(x,t) = \sum_{k=0}^{\infty} a_k \frac{\Psi_k(x)}{\|\Psi_k\|} e^{-T\lambda_k t}, \quad (x,t) \in \mathbb{R}\times[0, \tfrac{1}{2}], \tag{2.8}$$

where $\{-\lambda_k, \frac{\Psi_k}{\|\Psi_k\|}\}_{k\geq 0}$ is the orthonormal basis corresponding to the spectral decomposition of L_ε^*, where $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$, and the Fourier coefficients a_k are obtained from the boundary condition $\mu(x,0) = \mu(-x,\frac{1}{2}), x \in \mathbb{R}$.

Here is the key observation opening the route towards finding local maxima of the SPA coefficient. The terms in the sum (2.8) decay in time exponentially fast with rates λ_k, and therefore the terms corresponding to larger eigenvalues contribute less than the ones belonging to the low lying eigenvalues. This underlines their key importance. Fortunately, in the case of a double well potential the following theorem holds.

Theorem 2.3 ('spectral gap') *In the limit of small noise, the following asymptotics holds:*

$$\begin{aligned}
\lambda_0 &= \lambda_0(\varepsilon) = 0, \ \text{and } \Psi_0 = e^{-\frac{2U_1}{\varepsilon}}, \\
\lambda_1 &= \lambda_1(\varepsilon) = \frac{1}{2\pi}\sqrt{U_1''(1)|U_1''(0)|}\cdot e^{-v/\varepsilon}(1+\mathcal{O}(\varepsilon)), \\
\lambda_2 &= \lambda_2(\varepsilon) \geq C > 0 \ \text{uniformly in } \varepsilon.
\end{aligned}$$

The result of Theorem 2.3 plays a crucial role in our analysis. There is a *spectral gap* between the first eigenvalue and the rest of the spectrum. Consequently, only the first two terms of (2.8) can have an essential contribution to the SPA coefficient η^X.

2.4 Asymptotics of the SPA coefficient

The following theorem gives the asymptotics of the first two Fourier coefficients a_0 and a_1 in the Floquet type expansion of the previous subsection.

Theorem 2.4

$$\begin{aligned}
a_0 &= \|\Psi_0\|, \\
a_1 &= \frac{\|\Psi_1\|}{\|\Psi_0\|^2}\cdot\frac{\langle\Psi_0(-\cdot),\Psi_1\rangle}{\|\Psi_1\|^2 - e^{-\frac{1}{2}T\lambda_1}\langle\Psi_1(-\cdot),\Psi_1\rangle} + r
\end{aligned}$$

where r vanishes in the limit of small noise and for $T \geq \exp\{(v+\delta)/\varepsilon\}$, δ being positive and sufficiently small.

Recall the definition (2.6) of the SPA coefficient. Denote

$$S^X(\varepsilon, T) = \int_0^{\frac{1}{2}} \mathbf{E}_\mu X_{Ts}^{\varepsilon,T}\cdot e^{2\pi i s}\, ds. \tag{2.9}$$

Then we identify $\eta^X = 4|S^X|^2$.

Theorem 2.5 *Let $T \geq \exp\{(v+\delta)/\varepsilon\}$ for δ positive and sufficiently small. Then the following expansion for S^X holds in the small noise limit $\varepsilon \to 0$*

$$S^X = \frac{1}{\pi i}b_0 + \frac{1}{\pi i - \frac{1}{2}\lambda_1 T}b_1 + r_1$$

where the rest term r_1 tends to zero and the coefficients are given by

$$b_0 = \frac{\int y\, e^{-\frac{2U_1(y)}{\varepsilon}}\, dy}{\int e^{-\frac{2U_1(y)}{\varepsilon}}\, dy},$$

$$b_1 = -\frac{1+e^{-\frac{1}{2}T\lambda_1}}{2} \cdot \frac{\int y\, \Psi_1(y)\, dy}{\int e^{-\frac{2U_1(y)}{\varepsilon}}\, dy} \cdot \frac{\langle \Psi_0(-\cdot), \Psi_1\rangle}{\|\Psi_1\|^2 - e^{-\frac{1}{2}T\lambda_1}\langle \Psi_1(-\cdot), \Psi_1\rangle}.$$

Finally,

$$\eta^X = b_0^2 \frac{4}{\pi^2} \frac{(\lambda_1 T)^2}{4\pi^2 + (\lambda_1 T)^2} + R. \tag{2.10}$$

where R tends to zero with ε.

Let us now study the *resonance* behaviour of the SPA coefficient η^X, i.e. investigate whether it has a local maximum in ε. We formulate the following Lemma which is obtained by application of Laplace's method of asymptotic expansion of singular integrals, see [10, 22] or also [24, 19].

Lemma 2.6 ('Laplace's method') *In the small noise limit, the following holds true:*

$$b_0 = -1 - \frac{1}{4}\frac{U_1^{(3)}(-1)}{U_1''(-1)^2}\varepsilon + \mathcal{O}(\varepsilon^2),$$
$$b_1 = -1 + \mathcal{O}(\varepsilon),$$

and consequently

$$b_0^2 = 1 + \frac{1}{2}\frac{U_1^{(3)}(-1)}{U_1''(-1)^2}\varepsilon + \mathcal{O}(\varepsilon^2), \tag{2.11}$$
$$(b_0 - b_1)^2 = \mathcal{O}(\varepsilon^2).$$

The following Theorem exhibits the defect of the notion of spectral power amplification for our diffusions in periodically and discontinuously switching potential states.

Theorem 2.7 *Let us fix δ positive and sufficiently small and $\Delta > v + \delta$. Let also $U_1(x) - 2U_1(-x) < v + V$ for all $x \in \mathbb{R}$ (no strong asymmetry!). Then for $T \to \infty$ and ε from the domain*

$$\frac{v+\delta}{\ln T} \le \varepsilon \le \frac{\Delta}{\ln T} \tag{2.12}$$

the following asymptotic expansion for the SPA coefficient holds:

$$\eta^X(\varepsilon, T) = \frac{4}{\pi^2}\left(1 + \frac{1}{2}\frac{U_1^{(3)}(-1)}{U_1''(-1)^2}\varepsilon\right) + \mathcal{O}\left(\frac{1}{\ln^2 T}\right).$$

This result has the following surprising consequences.

Corollary 2.8 *For $T \to \infty$ and $\varepsilon \in [\frac{v+\delta}{\ln T}, \frac{\Delta}{\ln T}]$ the SPA coefficient is a decreasing function of ε if $U_1^{(3)}(-1) < 0$ and an increasing function of ε if $U_1^{(3)}(-1) > 0$.*

Thus, the SPA coefficient as quality measure for tuning shows *no resonance* in a domain above Freidlin's threshold for quasi-deterministic periodicity (Theorem 2.2). This contradicts the physical intuition for the 'effective dynamics'. The reason for this surprising phenomenon can only be hidden in the *intrawell* behaviour of the diffusion neglected when passing to the reduced Markov chain. We return to this question later. Let us next study mathematically the 'effective dynamics' of the diffusion (2.2).

2.5 The 'effective dynamics': two-state Markov chain

The idea of approximation of diffusions in potential landscapes by appropriate finite state Markov chains in the context of stochastic resonance was suggested by Eckmann and Thomas [9], and C. Nicolis [21], and developed by McNamara and Wiesenfeld [20]. In this section we follow [24, 19]. The discrete time case was studied in [18].

In order to catch the main features of the spatial *interwell* hoppings of the diffusion (2.2) we consider the time inhomogeneous Markov chain $Y^{\varepsilon,T}$ living on the diffusion's meta-stable states ± 1. The infinitesimal generator of $Y^{\varepsilon,T}$ is periodic in time and is given by

$$Q_{\varepsilon,T}(t) = \begin{cases} \begin{pmatrix} -\varphi & \varphi \\ \psi & -\psi \end{pmatrix}, & \frac{t}{T} \pmod 1 \in [0, \frac{1}{2}), \\ \begin{pmatrix} -\psi & \psi \\ \varphi & -\varphi \end{pmatrix}, & \frac{t}{T} \pmod 1 \in [\frac{1}{2}, 1). \end{cases} \tag{2.13}$$

The transition rates φ and ψ which are responsible for the similarity of the two processes are chosen to be exponentially small in ε:

$$\varphi = \frac{1}{2\pi}\sqrt{U_1''(-1)|U_1''(0)|} \cdot e^{-V/\varepsilon} \text{ and } \psi = \frac{1}{2\pi}\sqrt{U_1''(1)|U_1''(0)|} \cdot e^{-v/\varepsilon}.$$

To exponential order they correspond (as they should) to the inverses of the Kramers' transition times (see Theorem 2.1). The invariant measure of $Y_{Tt}^{\varepsilon,T}$ can be obtained as a solution of a forward Kolmogorov equation and is given by

$$\begin{aligned} \nu^-(t) &= \frac{\psi}{\varphi+\psi} + \frac{\varphi-\psi}{\varphi+\psi}\frac{e^{-(\varphi+\psi)Tt}}{1+e^{-\frac{1}{2}(\varphi+\psi)Tt}}, \\ \nu^+(t) &= \frac{\varphi}{\varphi+\psi} - \frac{\varphi-\psi}{\varphi+\psi}\frac{e^{-(\varphi+\psi)Tt}}{1+e^{-\frac{1}{2}(\varphi+\psi)Tt}}, \quad t \in [0, \tfrac{1}{2}], \end{aligned} \tag{2.14}$$

and $\nu^\pm(t) = \nu^\mp(t + \frac{1}{2})$ for $t \geq 0$.

We define the SPA coefficient η^Y for the Markov chain $Y^{\varepsilon,T}$ analogously to (2.6). In this much simpler setting given it can be described explicitly.

Theorem 2.9 *For all $\varepsilon > 0$ and $T > 0$ the following holds:*

$$\eta^Y(\varepsilon, T) = \frac{4}{\pi^2}\frac{T^2(\varphi-\psi)^2}{4\pi^2 + T^2(\varphi+\psi)^2}. \tag{2.15}$$

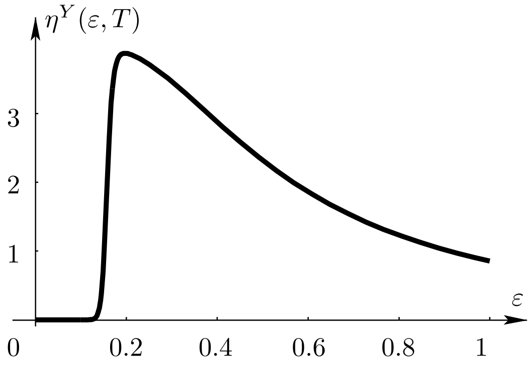

Fig. 8: SPA coefficient η^Y of the two-state Markov chain.

Compare (2.15) with (2.10). Since $(\varphi \pm \psi)^2 \approx \lambda_1^2$ in the limit of small ε, the formulae for η^X and η^Y differ only in the 'geometric' pre-factor b_0^2 and the asymptotically negligible rest term R.

The explicit formula (2.15) allows to study the local maxima of η^Y as a function of noise intensity for large periods T in great detail (see Fig. 8).

Theorem 2.10 *In the limit $T \to \infty$ the function $\varepsilon \mapsto \eta^Y(\varepsilon, T)$ has a local maximum at*

$$\varepsilon(T) \approx \frac{v+V}{2} \frac{1}{\ln T},$$

or in terms of T

$$T(\varepsilon) \approx \frac{\pi}{\sqrt{2pq}} \sqrt{\frac{v}{V-v}} e^{\frac{V+v}{\varepsilon}}.$$

The 'resonance' behaviors of η^X and η^Y are quite different. Whereas the diffusion's SPA has no extremum for small ε, the Markov chain's *always* has. What can be responsible for this discrepancy? Note that the Markov chain mimicks only the *interwell* dynamics of the diffusion. Thus, the SPA coefficient η^Y measures only the spectral energy contributed by interwell jumps. On the other hand, η^X also counts the numerous *intrawell* fluctuations of the diffusion the weight of which evidently becomes overwhelming in the small noise limit. These fluctuations have small energy. But since the diffusion spends most of its time near ± 1 the local asymmetries of the potential at these points dominate the picture and destroy optimal tuning.

To underpin this heuristics mathematically, let us now make the idea of neglecting the diffusion's intrawell fluctuations precise. For example, we *cut off* those among them which have not enough energy to reach half the height of the potential barrier between the wells. Consider the cut-off function g defined by

$$g(x) = \begin{cases} -1, & x \in [x_1, x_2], \\ 1, & x \in [y_1, y_2], \\ x, & \text{otherwise}, \end{cases}$$

where $x_1 < -1 < x_2 < 0$ and $0 < y_1 < 1 < y_2$ are such that $U_1(x_1) = U_1(x_2) = -\frac{V}{4}$ and $U_1(y_1) = U_1(y_2) = -\frac{v}{4}$, see Fig. 9. Now we study the modified SPA coefficient of

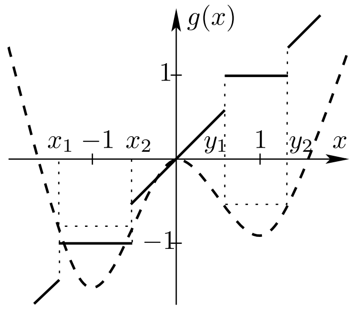

Fig. 9: Function g designed to cut off diffusion's intrawell dynamics.

a diffusion defined by

$$\widetilde{\eta}^X(\varepsilon, T) = \left| \int_0^1 \mathbf{E}_\mu \left[g(X_{Ts}^{\varepsilon,T}) \right] e^{2\pi i s}\, ds \right|^2 .$$

Following the steps of Subsection 2.4 we obtain a formula for $\widetilde{\eta}^X$ which is quite similar to (2.10) and (2.15):

$$\widetilde{\eta}^X(\varepsilon, T) = \widetilde{b}_0^2 \frac{4}{\pi^2} \frac{(\lambda_1 T)^2}{4\pi^2 + (\lambda_1 T)^2} + \widetilde{R},$$

where $\widetilde{R}$ is a small remainder term, and

$$\widetilde{b}_0^2 = \left(\frac{\int g(y) e^{-\frac{2U_1(y)}{\varepsilon}}\, dy}{\int e^{-\frac{2U_1(y)}{\varepsilon}}\, dy} \right)^2 = 1 - 4\sqrt{\frac{U_1''(-1)}{U_1''(1)}} e^{-\frac{V-v}{\varepsilon}} (1 + \mathcal{O}(\varepsilon))$$

(compare to (2.11)).

The modified geometric pre-factor $\widetilde{b}_0^2$ is essentially smaller than its counterpart b_0^2. This has crucial influence on the SPA coefficient $\widetilde{\eta}^X$: in the limit of large period and small noise its behavior now reminds of η^Y.

Theorem 2.11 *Let the assumptions of Theorem 2.7 hold. Then for any $\gamma > 1$ in the limit $T \to \infty$ the function $\varepsilon \mapsto \widetilde{\eta}^X(\varepsilon, T)$ has a local maximum on*

$$\left[\frac{1}{\gamma} \frac{v + V}{2} \frac{1}{\ln T}, \gamma \frac{v + V}{2} \frac{1}{\ln T} \right].$$

In other words, the optimal tuning for the measure of goodness $\widetilde{\eta}^X$ exists and is given approximately by

$$\varepsilon(T) \approx \frac{v + V}{2} \frac{1}{\ln T}.$$

2.6 Two-state Markov chain: different measures of quality of tuning

In this section we consider further measures of periodic tuning of the random trajectories of the two-state Markov chain $Y^{\varepsilon,T}$ with generator (2.13). These measures are based on the analysis of the invariant law of $Y^{\varepsilon,T}$ determined in (2.14). Due to the simplicity of the stochastic processes we face, they can be explicitly calculated, and their maxima resp. minima can be obtained rather easily. As seen above, Theorems 2.2 and 2.13 provide lower bounds for the exponential rates $T(\varepsilon)$ above which in the small noise limit $\varepsilon \to 0$ the randomly perturbed system produces trajectories with periodicity properties. Our results for the Markov chain all agree with this bound and determine the exact rates of growth for $T(\varepsilon)$ as $\varepsilon \to 0$ along with pre-factors. Given the discussion just completed, it should not be very surprising that they lead to critical tunings of optimal response, i.e. resonance points, which differ from the SPA resonance point, and eventually even differ among themselves. These measures have not yet been investigated in the diffusion framework. A comparison with their diffusion versions will very likely produce similar discrepancies as for the SPA coefficients discussed. For a detailed exposition of the results including their elementary proofs see Pavlyukevich [24].

2.6.1 Signal-to-Noise Ratio

A most prominent physical quality measure is known under the name *Signal-to-Noise Ratio (SNR).* It is defined an the ratio of spectral energy carried by the random output to energy carried by noise. Formally, it is given by

$$\mathrm{SNR}(\varepsilon,T) = \frac{1}{\varepsilon^2}\left|\int_0^1 \mathbf{E}_\nu(Y^{\varepsilon,T}_{Ts})\cdot e^{2\pi i s}\,ds\right|^2 = \frac{\eta^Y(\varepsilon,T)}{\varepsilon^2}.$$

In the following Theorem we describe the local maximum of SNR in ε.

Theorem 2.12 *In the limit $T\to\infty$ the function $\varepsilon \mapsto \mathrm{SNR}(\varepsilon,T)$ has a local maximum at*

$$\varepsilon(T) \approx \frac{v}{\ln T},$$

or in terms of T

$$T(\varepsilon) \approx \frac{\pi\sqrt{v}}{q\sqrt{\varepsilon}} e^{\frac{v}{\varepsilon}}.$$

As we see, for the SNR the optimal noise level is smaller.

2.6.2 Out-of-Phase measure

Another interesting quality measure is obtained by taking an averaged variant of Freidlin's measure considered in Theorem 2.2. Indeed, the approach by Freidlin [11] is valid for a quite general class of stochastic systems. Theorem 2.2 describes the quasi-deterministic periodic response of the diffusion. In this theorem periodic tuning is quantified by the Lebesgue measure of the time spent by the trajectory outside of some neighborhood of the minimum of the deep well. Hence in case of the Markov chain it describes the total amount of time spent in the 'wrong' place.

Theorem 2.13 ([11]) *For $\varepsilon > 0$, let the period $T = T_\varepsilon$ be such that*

$$\lim_{\varepsilon\to 0} \varepsilon \ln T_\varepsilon = \omega > 0,$$

let the function $\phi(t)$ be defined by (2.5), and Λ denote Lebesgue measure on $[0,1]$. Then, if $\omega > v$,

$$\Lambda(t \in [0,1] \,:\, Y^{\varepsilon,T}_{T_\varepsilon t} \neq \phi(t)) \to 0 \tag{2.16}$$

in probability as $\varepsilon \to 0$.

We now consider the Lebesgue measure given by (2.16) as a function of ε and T

$$\Lambda(\varepsilon, T) = \Lambda(t \in [0,1] : Y^{\varepsilon,T}_{Tt} \neq \phi(t)).$$

Note that $\Lambda(\varepsilon, T)$ is a random variable. We introduce the *out-of-phase measure* by

$$d(\varepsilon, T) = \mathbf{E}_\nu \Lambda(\varepsilon, T) = \mathbf{E}_\nu \int_0^1 \mathbf{I}(Y^{\varepsilon,T}_{Ts} \neq \phi(s))\, ds.$$

The out-of-phase measure describes how much time on average the Markov chain spends in the 'wrong' state.

Theorem 2.14 *a) The out-of-phase measure is given by*

$$d(\varepsilon, T) = \frac{1}{\varphi + \psi}\left[\varphi - \frac{1}{T}\frac{\varphi - \psi}{\varphi + \psi}\tanh\frac{T(\varphi + \psi)}{2}\right].$$

b) For T large enough, $\varepsilon \mapsto d(\varepsilon, T)$ has a local minimum. The optimal tuning rate at which this minimum is attained is given by

$$\varepsilon(T) \approx \frac{V}{\ln T},$$

or in terms of T

$$T(\varepsilon) \approx \frac{1}{p}\frac{v}{V - v} e^{\frac{V}{\varepsilon}}$$

Remark 2.15 The out-of-phase measure can be rewritten as

$$d(\varepsilon, T) = \frac{1}{4}\int_0^1 \mathbf{E}_\nu (Y^{\varepsilon,T}_{Ts} - \phi(s))^2\, ds.$$

This expression represents the mean square deviation of $Y^{\varepsilon,T}$ from the deterministic function ϕ describing the energetically most favorable place in the potential landscape as a function of time.

2.6.3 Relative entropy

Let us introduce the time dependent Dirac mass $\delta_{\phi(t)}$ in $\phi(t)$ for $t \geq 0$, with ϕ according to (2.5). We have $\delta_{\phi(t)} = \phi^-(t)\delta_{-1} + \phi^+(t)\delta_1$, where $\phi^-(t) = 1$, $\phi^+(t) = 0$ if $t \pmod 1 \in [0, \frac{1}{2})$ and $\phi^-(t) = 0$, $\phi^+(t) = 1$ if $t \pmod 1 \in [\frac{1}{2}, 1)$. The measure thus defined has full mass in the position of the minimum of the deep well as a function of time.

The notion of quality of periodic tuning we next consider is the *relative entropy* of the invariant measure $\nu_{\varepsilon,T}$ with respect to δ_ϕ, and is formally given by

$$H_{\phi|\nu}(\varepsilon, T) = \int_0^1 \sum_{\alpha=+,-} \phi^\alpha(s) \ln \frac{\phi^\alpha(s)}{\nu^\alpha_{\varepsilon,T}(s)}\, ds.$$

The point of optimal tuning for relative entropy is given by minimization this time, and corresponds to exponential order to the one described by the Out-of-Phase measure.

Theorem 2.16 *a) The relative entropy is explicitly given by*

$$H_{\phi|\nu}(\varepsilon, T) = -\ln \frac{\psi}{\varphi + \psi} +$$
$$\frac{1}{T(\varphi+\psi)} \left[\mathrm{Li}_2 \left(\frac{\psi - \varphi}{\psi} \frac{e^{-T(\varphi+\psi)}}{1 + e^{-T(\varphi+\psi)}} \right) - \mathrm{Li}_2 \left(\frac{\psi - \varphi}{\psi} \frac{1}{1 + e^{-T(\varphi+\psi)}} \right) \right],$$

where $\mathrm{Li}_2(x)$ *is the dilogarithm function defined by* $\mathrm{Li}_2(x) = \int_x^0 \frac{\ln(1-y)}{y}\, dy$, $x \leq 1$.

b) For T large enough, $\varepsilon \mapsto H_{\phi|\nu}(\varepsilon, T)$ has a local minimum, and an optimal tuning rate is given by

$$T(\varepsilon) \approx \frac{\pi^2}{6p} \frac{v}{V - v} e^{\frac{V}{\varepsilon}}$$

or in terms of ε

$$\varepsilon(T) \approx \frac{V}{\ln T}.$$

2.6.4 Entropy

The *entropy* of the invariant measure $\nu^\pm_{\varepsilon,T}$ at time $t \in [0, 1]$ is given by

$$H_t(\varepsilon, T) = -\nu^-_{\varepsilon,T}(t) \ln \nu^-_{\varepsilon,T}(t) - \nu^+_{\varepsilon,T}(t) \ln \nu^+_{\varepsilon,T}(t), \quad t \in [0, 1].$$

With its help we may define the *entire entropy* of $\nu^\pm_{\varepsilon,T}$ by

$$H(\varepsilon, T) = \int_0^1 H_s(\varepsilon, T)\, ds.$$

The following Theorem quantifies the entire entropy explicitly, and asymptotically characterizes its minimum as the point of optimal tuning corresponding to this measure.

Theorem 2.17 *a) The entire entropy of $\nu_{\varepsilon,T}$ is given by (see Fig. 10)*

$$
\begin{aligned}
H(\varepsilon,T) &= \ln(\varphi+\psi) - \frac{\varphi\ln\varphi+\psi\ln\psi}{\varphi+\psi} \\
&+ \frac{\varphi}{T(\varphi+\psi)^2}\left[\mathrm{Li}_2\left(\frac{\varphi-\psi}{\varphi}\frac{1}{1+e^{-T(\varphi+\psi)}}\right) - \mathrm{Li}_2\left(\frac{\varphi-\psi}{\varphi}\frac{e^{-T(\varphi+\psi)}}{1+e^{-T(\varphi+\psi)}}\right)\right] \\
&+ \frac{\psi}{T(\varphi+\psi)^2}\left[\mathrm{Li}_2\left(\frac{\psi-\varphi}{\psi}\frac{1}{1+e^{-T(\varphi+\psi)}}\right) - \mathrm{Li}_2\left(\frac{\psi-\varphi}{\psi}\frac{e^{-T(\varphi+\psi)}}{1+e^{-T(\varphi+\psi)}}\right)\right],
\end{aligned}
$$

where $\mathrm{Li}_2(x)$ *is the dilogarithm function,* $\mathrm{Li}_2(x) = \int_x^0 \frac{\ln(1-y)}{y}\,dy$, $x \le 1$.

b) For T large enough $H(\varepsilon,T)$ always has a local minimum, and an optimal tuning rate is given by

$$T(\varepsilon) \approx \frac{\pi^2}{6p}\frac{v\varepsilon}{(V-v)^2}e^{\frac{V}{\varepsilon}},$$

or in terms of ε

$$\varepsilon \approx \frac{V}{\ln T}.$$

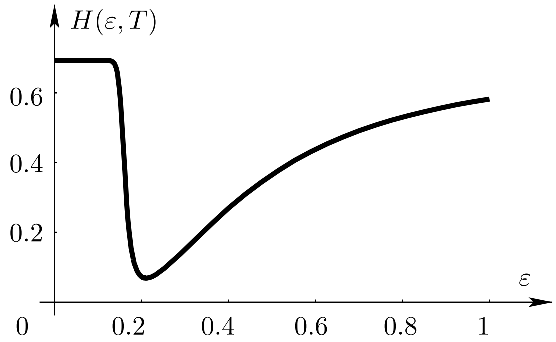

Fig. 10: The entire entropy of the invariant measure of the two-state Markov chain as a function of noise intensity.

The measures of quality studied here can be subdivided into two groups. The SPA coefficient and the SNR belong to the first group. They are based on the interpretation of the Markov chain as a random amplifier and depend on the spectral properties of the averaged output. The key result which may seem counterintuitive is as follows: an optimal transfer of a deterministic periodic signal through a random system is not guaranteed by the elimination of noise, but rather by tuning it in on some essentially non-zero intensity level.

The same result holds for the second group of measures which can be seen as measures of stabilization. Indeed, the out-of-phase measure and the relative entropy determine the deviation of the random output from a deterministic function ϕ. Again, *increasing* the noise makes the random output *less random*. The final measure — the entropy of the invariant measure of the process — is, roughly speaking, the measure of randomness. The bigger the entropy, the more chaotic is the system. The fact that non-zero noise minimizes the entropy is a very good example of noise-induced order.

It is interesting to note that if the exponential rate is v, i.e. corresponds to the depth of the shallow well, then the pre-factors contain only the parameter q, which stands for the geometry of the shallow well. Analogously, if $T(\varepsilon) \sim e^{V/\varepsilon}$, then the pre-factor of the optimal tuning rate only contains p, the geometrical factor of the deep well, and not q. The only exception is the SPA coefficient η^Y whose optimal tuning rate contains all parameters of the system.

3 Smooth periodic potentials and a robust resonance notion

The serious defect of the SPA coefficient in the prediction of the stochastic resonance point in the Markov chain models containing the effective dynamics of complex diffusion models motivated us to look for *robust* notions of quality of periodic tuning. Since the dynamics of the Markov chain only retains the rough mechanism of transitions between the domains of attraction given in the underlying potential landscape, such a notion should only take into account the most important aspects of the *attractor hopping*. Also, as the alternative notions discussed in the preceding section show, the resonance point is by no means a canonical object, independent of the way tuning quality is measured. We think that the methods of advanced large deviations' theory behind the notion to be explained in this section will give it a more natural place, and possibly qualify it as canonical.

At the same time, we essentially generalize the simplified model of time periodic potential considered in the previous section, and lift the study of stochastic resonance to a somewhat more abstract level. The potential function U in the present section will still be supposed to be one-dimensional in space. But its periodic time variation will just be assumed to be continuous, and otherwise quite general. More precisely, we study diffusion processes driven by a Brownian motion of intensity ε given by the stochastic differential equation

$$dX_t^{\varepsilon,T} = -U'(X_t^{\varepsilon,T}, \frac{t}{T})\, dt + \sqrt{\varepsilon}\, dW_t, \quad t \geq 0.$$

The underlying potential landscape (see Fig. 11) is described by a function $U(x,t)$, $x \in \mathbb{R}$, $t \geq 0$, which is periodic in time with period 1, and its temporal variation, by the rescaling with very large T, acts on the diffusion at a very small frequency. U is supposed to have exactly two wells located at ± 1, separated by a saddle at 0. The depth of $U(\cdot, t)$ at ± 1 is given by the 1-periodic depth functions $\frac{1}{2} D_{\pm 1}(t)$ which are assumed to never fall below zero. If we look at time scales $T = e^{\omega/\varepsilon}$, Freidlin's theory of quasi-deterministic motion explained in Section 2.1 indicates that transitions e.g. from the domain of attraction of -1 to the domain of attraction of 1 will occur as soon

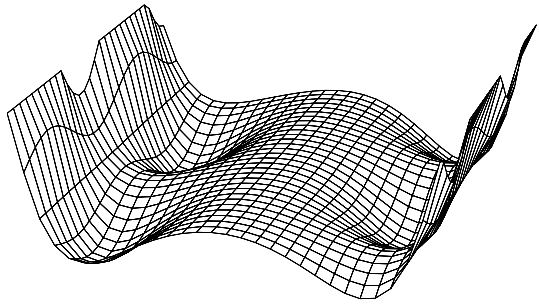

Fig. 11: Potential landscape U.

as D_{-1} gets less than ω, i.e. at time

$$a_\omega^{\pm 1} = \inf\{t \geq 0 : D_{\pm 1}(t) \leq \omega\}.$$

This triggers periodic behavior of the diffusion trajectories on long time scales. The modern theory of meta-stability in *time homogeneous diffusion processes* produces the exponential decay rates of transition probabilities between different domains of attraction of a potential landscape together with very sharp multiplicative error estimates, uniformly on compacts in system parameters. Their sharpest forms are presented in some very recent papers by Bovier et al. [5, 6]. We use this powerful machinery in order to obtain very precise estimates of the exponential tails of the laws of the transition times between domains of attraction. To this end, we have to extend the estimates by Bovier et al. [6] to the framework of *time inhomogeneous diffusions*. In the underlying one-dimensional situation, this can be realized by freezing the time dependence of the potential on small time intervals to define lower and upper bound time homogeneous potentials not differing very much from the original one. Comparison theorems are used to control the transition behavior from above and below through the corresponding time homogeneous diffusions. This allows very precise estimates on the probabilities with which the diffusion at time scale $T = e^{\omega/\varepsilon}$ transits from the domain of attraction of -1 to the domain of attraction of 1 or vice versa within time windows $[(a_\omega - h)T, (a_\omega + h)T]$ for small $h > 0$. If $\tau_x(X^{\varepsilon,T})$ denotes the transit time to x, it is given by

$$\lim_{\varepsilon\to 0} \varepsilon \ln\left(1 - M(\varepsilon,\omega)\right) = \max_{i=\pm 1}\left\{\omega - D_i(a_\omega^i - h)\right\},$$

with

$$M(\varepsilon,\omega) = \min_{i=\pm 1} P_i(\tau_{-i}(X^{\varepsilon,T}) \in [(a_\omega^i - h)T, (a_\omega^i + h)T]), \quad \varepsilon > 0, \omega \in I_R,$$

and where I_R is the *resonance interval*, i.e. the set of scale parameters for which trivial or chaotic transition behavior of the trajectories is excluded. The stated convergence is *uniform* in ω on compact subsets of I_R. This allows us to take $M(\varepsilon,\omega)$ as our measure of periodic tuning, compute the scale $\omega_0(h)$ for which the transition rate is optimal, and define the *stochastic resonance point* as the eventually existing limit of $\omega_0(h)$ as $h \to 0$. This resonance notion has the big advantage of being robust for the passage from the diffusion to the two state Markov chain describing the effective dynamics.

3.1 Transition times for the Markov chain

Let us first discuss the effective dynamics modelled by a continuous time two state Markov chain. The states represent the positions of the bottoms of the wells of the double well potential. The transition rates picture the transition mechanism of the diffusion to which we return later. We shall first define the interval of time scales for which transitions are not trivial.

3.1.1 Definition of the resonance interval

Let us consider the time continuous Markov chain $Y^{\varepsilon,T} = (Y_t^{\varepsilon,T})_{t\geq 0}$ taking values in the state space $\{-1, 1\}$ with initial data $Y_0^{\varepsilon,T} = -1$. Suppose that the infinitesimal generator is given by

$$G_{\varepsilon,T}(t) = \begin{pmatrix} -\varphi(\frac{t}{T}) & \varphi(\frac{t}{T}) \\ \psi(\frac{t}{T}) & -\psi(\frac{t}{T}) \end{pmatrix},$$

where T is an exponentially large time scale (we assume that $T = e^{\omega/\varepsilon}$, $\omega > 0$), ψ and φ are 1-periodic functions describing a rate which just produces the transition dynamics of the diffusion between the potential minima ± 1. Let us recall that, if we consider some time-independent potential U then the mean transition time between the wells is given by Kramers' law. If the diffusion starts in the minimum of one well, the mean exit time is equivalent to $e^{V/\varepsilon}$, where $\frac{V}{2}$ is the height of the barrier separating the two minima of the potential. Consequently the transition rate should be proportional to $e^{-V/\varepsilon}$.

In the framework we now consider the depth of the wells depends continuously on time. In this situation it is natural to consider the following periodic infinitesimal probabilities

$$\varphi(t) = e^{-D_{-1}(t)/\varepsilon}, \quad t \geq 0. \tag{3.1}$$

Let us assume that $D_1(t) = D_{-1}(t + \alpha)$, $t \geq 0$, with phase shift $\alpha \in]0, 1[$ and

- all local extrema of $D_{\pm 1}(\cdot)$ are global;
- the functions $D_{\pm 1}(\cdot)$ are strictly monotonous between the extrema.

Hence $\psi(t) = \varphi(t + \alpha)$, $t \geq 0$, and

$$\psi(t) = e^{-D_1(t)/\varepsilon}, \quad t \geq 0. \tag{3.2}$$

Let us define S_{-1} to be the normalized time of the first jump from the state -1 to 1, i.e. $S_{-1} = \inf\{t \geq 0 : Y_{tT}^{\varepsilon,T} = 1\}$ Analogously, S_1 will be the time of first jump from state 1 to -1, starting with $Y_0 = 1$. We are especially interested in the behavior of S as T becomes very large, that is as $\varepsilon \to 0$. In fact we get the following dichotomy of possible behavior:

- If $\omega > \inf_{t\geq 0} D_{-1}(t)$, the law of S_{-1} tends to the Dirac measure in the point a_ω^{-1} given by

$$a_\omega^{\pm 1} = \inf\{t \geq 0 : D_{\pm 1}(t) \leq \omega\}. \tag{3.3}$$

- If $\omega \le \inf_{t\ge 0} D_{-1}(t)$, then the probability measure of S_{-1} tends weakly to the null measure.

It suffices to replace D_{-1} by D_1 and a_ω^{-1} by a_ω^1 to obtain similar results for S_1.

This leads to the following interpretation:

If $\omega \ge D_{-1}(0)$, that is, if the time scale T is very large, then on this exponential scale, the asymptotic behavior of the Markov chain is characterized by an instantaneous jump, i.e. $a_\omega^{-1} = 0$. This just means that a clock ticking in units of T will record a jump of the process as instantaneous, since it occurs on a smaller scale.

In case $\omega < \inf_{t\ge 0} D_{-1}(t)$, the time scale T is too small compared to the transition rates. Consequently no transitions will be observed, and the process never jumps on this scale.

In the last case $D_{-1}(0) > \omega > \inf_{t\ge 0} D_{-1}(t)$. So the infinitesimal probability at time 0 is too small to allow any transition, and the Markov chain will have to wait until this probability is small enough to allow for jumps, that is approximatively $a_\omega T$. This case is the only interesting case, in the sense that the chain stays for some time in the starting state before it jumps to the other one.

To observe stochastic resonance we obviously need to study both transitions from -1 to 1 and vice versa. So we define some interval I_R called *interval of resonance* (see Fig. 12) which is to contain those exponential scales in which the process on the one hand asymptotically cannot always stay in the same state with positive probability, and on the other hand asymptotically cannot jump instantaneously from one state to the other.

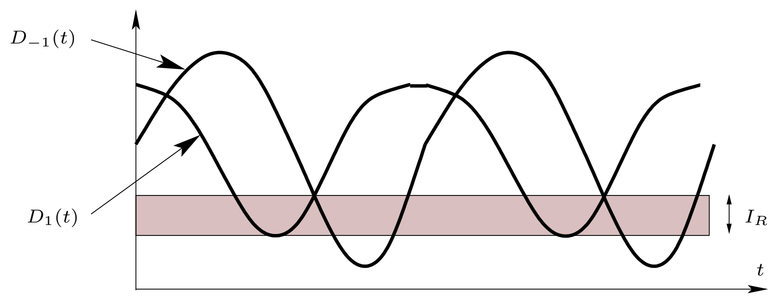

Fig. 12: Resonance interval I_R.

$$I_R =]\max_{i=\pm 1} \inf_{t\ge 0} D_i(t), \inf_{t\ge 0} \max_{i=\pm 1} D_i(t)[.$$

3.1.2 Optimal tuning for the Markov chain

Let us now assume that we are in the range of non-trivial jumping, that is $\omega \in I_R$. We next determine an optimal tuning rate or stochastic resonance point. It will be based on the density of the first jump, in particular the intensity of its peak, which we propose as a new measure of quality of tuning. For $h > 0$ we shall compare the probabilities with which the first transition takes place within the window of exponential length

$[(a^i_\omega - h)T, (a^i_\omega + h)T]$, $i = \pm 1$, for different ω, maximize this quantity in ω and finally take the window length to 0. More formally, for $h > 0$ small enough define

$$N(\varepsilon, \omega) = \min_{i=\pm 1} \mathbf{P}_i(S_i \in [(a^i_\omega - h)T, (a^i_\omega + h)T]), \ \varepsilon > 0, \ \omega \in I_R, \tag{3.4}$$

and call it *transition probability for a time window of width h* for the Markov chain. The optimal parameter ω_0 will tell us at which time scale it is most likely to see trajectories of the chain with first jump in the corresponding window, and further jumps in accordingly displaced windows. In particular, it will tell us at which scale periodic trajectories of just this period are most probable. Since the probability density of the first transition times from one state to the other is well known, for example the density of S_{-1} equals

$$p(t) = \varphi(t) e^{-\int_0^t \varphi(s) ds},$$

we can compute an explicit expression for $N(\varepsilon, \omega)$. The optimal time scale will be determined by a combination of a large deviations result concerning the first jump of the Markov chain parametrized by the logarithmic scale ω of time, and a maximization problem for the large deviation rates in ω to which the transition probabilities converge uniformly.

Using Laplace's method to estimate the singular integrals appearing as $\varepsilon \longrightarrow 0$, we obtain the required asymptotic result.

Theorem 3.1 *Let Γ be a compact subset of I_R, $h_0 < \max\{a^{-1}_\omega, \frac{T}{2} - a^{-1}_\omega\}$. Then for $0 < h \leq h_0$*

$$\lim_{\varepsilon \to 0} \varepsilon \ln(1 - N(\varepsilon, \omega)) = \max_{i=\pm 1} \left\{\omega - D_i(a^i_\omega - h)\right\} \tag{3.5}$$

uniformly for $\omega \in \Gamma$.

Since the convergence is uniform in ω, it suffices to minimize the left hand side of (3.5) to obtain an optimal tuning point. For h small the eventually existing global minimizer $\omega_R(h)$ of

$$I_R \ni \omega \mapsto \max_{i=\pm 1} \left\{\omega - D_i(a^i_\lambda - h)\right\}$$

is a good candidate for our resonance point. But it still depends on h. To get rid of this dependence, we shall consider the limit of $\lambda_R(h)$ as $h \to 0$.

Definition 3.2 *Suppose that*

$$I_R \ni \omega \mapsto \max_{i=\pm 1} \left\{\omega - D_i(a^i_\omega - h)\right\}$$

possesses a global minimum $\omega_R(h)$. Suppose further that

$$\omega_R = \lim_{h \to 0} \omega_R(h)$$

exists in I_R. We call ω_R the stochastic resonance point *of the Markov chain $Y^{\varepsilon,T}$ with time periodic infinitesimal generator $G_{\varepsilon,T}$.*

In fact the stochastic resonance point exists if one of the depth functions, and therefore both, due to the phase lag, has a unique point of maximal decrease in the interval in which it is strictly decreasing.

Example: In fact all the results presented before, in the case of a time dependent potential U with meta-stable states at ± 1 also hold true if the meta-stable states are allowed to move periodically but stay away from the saddle 0. Then the state -1 of the Markov chain represents the left meta-stable state and 1 represents the right one. We shall mention one classical example in stochastic resonance (see, for instance [13]) which is the over-damped motion of a Brownian particle in the potential

$$2U(x,t) = V(x) + Ax\cos(2\pi t),$$

where V denotes a reflection-symmetric potential with two wells located at ± 1. In this particular case, for $0 < A < V(0) - V(-1)$,

$$D_{\pm 1}(t) = V(0) - V(-1) \pm A\cos(2\pi t).$$

Hence the phase lag α is equal to π and the resonance interval is

$$I_R =]V(0) - V(-1) - A, V(0) - V(-1)[.$$

Let $h > 0$ small enough, then the logarithmic time scale which asymptotically optimizes the quality measure $N(\varepsilon, \omega)$ is given by

$$\omega_R(h) = V(0) - V(-1) - A\sin(\pi h).$$

In order to obtain the resonance point, we just let h tend to zero, to obtain $\omega_R = \lim_{h\to 0}\omega_R(h) = V(0) - V(-1)$, that is the average depth of the time periodic potential U. In this particular case, it is obvious that the resonance point coincides with the point of maximal decrease of the depth functions D_{-1} and D_1. This example is treated in detail in [15].

3.2 Transition times for the diffusion and robustness

As seen in the preceding subsection, for the effective dynamics we obtain both simple and explicit results. Now we shall show how our measure of quality based purely on the jumps for the two-state Markov chain can be extended to the diffusion case. We just have to generalize the notion of jumps to the transition times between the two domains of attraction of the potential landscape, i.e. the two wells. The accordingly generalized measure od quality of periodic tuning possesses the desired property of being robust. The analogous notion of interval of resonance will then be the following.

3.2.1 Resonance interval for diffusions

Recall that the underlying potential is described by a function $U(x,t)$, $x \in \mathbb{R}$, $t \geq 0$, such that $U'(\cdot,\cdot)$ is both continuous in time and space. The local minima are located at ± 1 and the saddle point at 0, independently of time. Our main concern will be the asymptotics of the transition times from the domain of attraction $]-\infty,0[$ of -1 to the domain of attraction $]0,\infty[$ associated with 1 of the time inhomogeneous diffusion in the small noise limit $\varepsilon \to 0$. More precisely we will be interested in describing the exponential transition rate from the domain of attraction of -1 to the domain of attraction of 1. Our potential not being time homogeneous, we shall make use of comparison arguments with diffusions possessing time independent potentials in order to perform a careful reduction of the inhomogeneous exit problem to the homogeneous one, and use the asymptotic results well known for this particular case. This will be achieved by freezing the driving force derived from the potential on small time intervals on the minimal or maximal level it takes there. To be more precise, for each interval $I \subset \mathbb{R}_+$ let

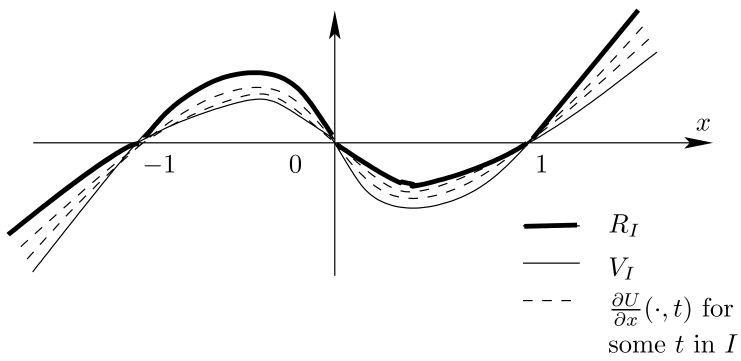

Fig. 13: Definition of V_I and R_I.

$$V_I(x) = \sup_{t\in I} \frac{\partial U}{\partial x}(t,x) \quad \text{and} \quad R_I(x) = \inf_{t\in I} \frac{\partial U}{\partial x}(t,x). \tag{3.6}$$

The regularity conditions valid for U imply that V and R are continuous functions. Moreover $V_I(-1) = R_I(-1) = 0$, see Fig. 13. If $I = [a,b]$, we denote by $\overline{X}^{\varepsilon,I}$ the solution of the SDE on $\mathbb{R}_+$

$$\begin{cases} d\overline{X}_t^{\varepsilon,I} &= -R_I(\overline{X}_t^{\varepsilon,I})\,dt + \sqrt{\varepsilon}\,dW_t, \\ \overline{X}_0^{\varepsilon,I} &= X_{aT}^{\varepsilon,T}. \end{cases} \tag{3.7}$$

$\underline{X}^{\varepsilon,I}$ is defined in the same way replacing R_I by V_I. These two *time homogeneous* diffusions are used to control the *time inhomogeneous* diffusion $X^{\varepsilon,T}$ as long as time runs in the interval I. In fact, we have P-a.s.

$$\underline{X}_{tT}^{\varepsilon,I} \le X_{(t+a)T}^{\varepsilon,T} \le \overline{X}_{tT}^{\varepsilon,I}, \quad t \in [0, b-a].$$

Hence in order to study the time the diffusion needs to reach 1 starting in the left well, we shall consider the diffusion on one period. This time interval can be decomposed into finitely many small time intervals I_n, $0 \le n \le n_0$. We shall then freeze the potential on I_n and analyze if the the diffusions $\underline{X}^{\varepsilon,I_n}$ and $\overline{X}^{\varepsilon,I_n}$ have enough time in I_n to reach the top of the barrier between the two wells and, consequently on the same scale reach 1, the bottom of the right well. In other words we need to get information on the exit problem for the homogeneous diffusions $\underline{X}^{\varepsilon,I}$ and $\overline{X}^{\varepsilon,I}$.

We shall refer to the most recent and advanced development of sharp estimates for transition times presented in Bovier et al. [5] and [6]. They are valid far beyond our modest framework, and we just present the results we will use here. For this purpose, suppose that $U_1(\cdot)$ is a purely space dependent $\mathcal{C}^2$ potential function of the shape similar to those on Fig. 5. It possesses only ± 1 as local minima, separated by the saddle 0. Suppose that the curvature of U_1 at -1 is strictly positive, i.e. $U_1''(-1) > 0$. As for ultra- or hypercontractivity type properties for U_1, we shall assume that it has exponentially tight level sets, i.e. there is $M_0 > 0$ such that for any $M \geq M_0$ there exists a constant $C(M)$ such that for $\varepsilon \leq 1$

$$\int_{\{y: U_1(y) \geq M\}} e^{-2U_1(z)/\varepsilon}\, dz < C(M) e^{-M/\varepsilon}. \tag{3.8}$$

We shall concentrate in this situation on an exit of the domain of attraction of the stable point -1 for the diffusion associated with the SDE

$$\begin{cases} dX_t^\varepsilon = -U_1'(X_t^\varepsilon)\, dt + \sqrt{\varepsilon}\, dW_t, \\ X_0^\varepsilon = x < 0. \end{cases}$$

We are interested in the asymptotics of the first time X^ε reaches 1:

$$\tau_1(X^\varepsilon) = \inf\{t > 0 : \ X_t^\varepsilon = 1\}.$$

Then we obtain the following result.

Theorem 3.3 *Let $\lambda(\varepsilon)$ denote the principal eigenvalue of the linear operator*

$$L_\varepsilon u = \frac{\varepsilon}{2} u'' - U_1' u', \quad u \in \mathcal{L}^2(]-\infty, 1], e^{-2U_1/\varepsilon} dx)$$

with Dirichlet boundary conditions at 1*. Then for every compact $K \subseteq]-\infty, 0[$ there is a constant $c > 0$ such that*

$$\mathbf{P}_x(\tau_1(X^\varepsilon) > t) = e^{-\lambda(\varepsilon) t}(1 + \mathcal{O}_K(e^{-c/\varepsilon})), \tag{3.9}$$

where $\mathcal{O}_K$ denotes an error term which is uniform in $x \in K$, $t \geq 0$. Moreover, for the asymptotic behavior of the eigenvalue $\lambda(\varepsilon)$ the following holds

$$\lambda(\varepsilon) \mathbf{E}_x[\tau_1(X^\varepsilon)] \to 1 \ \text{ uniformly on compacts } K \subseteq]-\infty, 0[\text{ as } \varepsilon \to 0. \tag{3.10}$$

Large deviations' theory reveals the asymptotic behavior of the principal eigenvalue: $\lim_{\varepsilon \to 0} \varepsilon \ln \lambda(\varepsilon) = -2(U_1(0) - U_1(-1))$. We deduce that the mean hitting time $\mathbf{E}_x[\tau_1(X^\varepsilon)]$ is equivalent to $e^{2(U_1(0) - U_1(-1))/\varepsilon}$ as $\varepsilon \to 0$. Here $U_1(0) - U_1(-1)$ is the depth of the starting well. Moreover, by Theorem 3.3, the normalized hitting time $\frac{\tau_1(X^\varepsilon)}{\mathbf{E}_x[\tau_1(X^\varepsilon)]}$ converges in law to an exponential random variable with mean 1 as $\varepsilon \to 0$.

These results are very precise. They describe the asymptotic time of the barrier crossing and at the same time give an estimation of the probability to cross the barrier in a small time window around this asymptotic deterministic time. We can apply them to the 'frozen' potential $U(\cdot,\cdot)$ on the small time intervals I_n. We thereby assume for simplicity that the frozen potentials are regular of order $\mathcal{C}^2$. Let us choose $n \geq 0$ and set $I_n = [r_n, r_{n+1}]$. We assume that $X^{\varepsilon,T}$ has not reached the top of the barrier before $r_n T$ and study what happens during the time interval $[r_n T, r_{n+1}T]$. We have already seen that $X^{\varepsilon,T}$ is controlled by both $\underline{X}^{\varepsilon,I_n}$ and $\overline{X}^{\varepsilon,I_n}$. On the one hand, it suffices to prove that $\underline{X}^{\varepsilon,I_n}$ reaches 1 before $r_{n+1}T$ in order to get $\tau_1(X^{\varepsilon,T}) \leq r_{n+1}T$. On the other hand, if we get that $\overline{X}^{\varepsilon,I_n}$ does not hit 1 then so does $X^{\varepsilon,T}$. As $\varepsilon \to 0$, Theorem 3.3 tells us, for example, that the probability that $\underline{X}^{\varepsilon,I_n}$ reaches 1 before $r_{n+1}T$ is close to 1 if the depth of the left well is smaller than $\lim_{\varepsilon\to 0} \varepsilon \ln(r_{n+1} - r_n)T = \omega$. Indeed we get $\lim_{\varepsilon\to 0}(r_{n+1} - r_n)\lambda(\varepsilon)T = +\infty$ which implies by (3.9) that

$$\lim_{\varepsilon\to 0} \mathbf{P}_x(\tau_1(\underline{X}^{\varepsilon,I_n}) > (r_{n+1} - r_n)T) = 0.$$

The statements depend weakly on the depth of the well of the potential associated to $\underline{X}^{\varepsilon,I_n}$ and $\overline{X}^{\varepsilon,I_n}$. Since $\frac{\partial U}{\partial x}$ is continuous both in x and t, if we choose the length of all intervals I_n small enough then the well depth functions associated with the two time homogeneous diffusions are equivalent to $D_{-1}(r_n)$, the depth of the left well of the landscape U. Hence the diffusion $X^{\varepsilon,T}_{tT}$ reaches 1 asymptotically as soon as the depth $D_{-1}(t)$ goes below the level ω. This means

$$\lim_{\varepsilon\to 0} \frac{\tau_1(X^{\varepsilon,T})}{T} = a_\omega^{-1},$$

where a_ω^{-1} was defined in (3.3).

Knowing the asymptotics of the time at which the diffusion reaches the barrier separating the two wells in order to hit 1 puts us again in a position in which we can discuss a resonance interval as for the reduced model. We obtain the same interval

$$I_R =]\max_{i=\pm 1} \inf_{t\geq 0} D_i(t), \inf_{t\geq 0} \max_{i=\pm 1} D_i(t)[.$$

3.2.2 Optimal tuning for the diffusion and robustness

The comparison between time inhomogeneous and homogeneous potentials and the asymptotic result 3.3 enable us to proceed to the completion of our approach of stochastic resonance for diffusions. We have very precise estimates on the probabilities with which the diffusion at time scale $T = e^{\omega/\varepsilon}$ transits from the domain of attraction of -1 to the domain of attraction of 1 and vice versa within the time windows $[(a_\omega^i - h)T, (a_\omega^i + h)T]$ for small $h > 0$. On their basis we may define a measure of quality of tuning for the diffusion which corresponds to (3.4):

$$M(\varepsilon,\omega) = \min_{i=\pm 1} \mathbf{P}_i(\tau_{-i}(X^{\varepsilon,T}) \in [(a_\omega^i - h)T, (a_\omega^i + h)T]), \quad \varepsilon > 0, \omega \in I_R, \tag{3.11}$$

We may now state our main result on uniform transition rates.

Theorem 3.4 *Let Γ be a compact subset of I_R, $h_0 > 0$ small enough. Then*

$$\lim_{\varepsilon \to 0} \varepsilon \ln(1 - M(\varepsilon, \omega)) = \max_{i=\pm 1} \left\{ \omega - D_i(a^i_\omega - h) \right\} \tag{3.12}$$

uniformly for $\omega \in \Gamma$.

The stated convergence is *uniform* in ω on compact subsets of I_R. This allows us to take $M(\varepsilon, \omega)$ as our measure of periodic tuning, compute the scale $\omega_0(h)$ for which the transition rate is optimal, and define the *stochastic resonance point* as the eventually existing limit of $\omega_0(h)$ as $h \to 0$. This notion of quality has the big advantage of being robust for the passage from the two state Markov chain to the diffusion. So the following final robustness result holds true.

Theorem 3.5 *The resonance points of the diffusion $X^{\varepsilon,T}$ with periodic potential U and of the Markov chain $Y^{\varepsilon,T}$ with exponential transition rate functions $D_{\pm 1}$ coincide.*

References

[1] V. S. Anishchenko, A. B. Neiman, F. Moss, and L. Schimansky-Geier. Stochastic resonance: noise-enhanced order. *Physics–Uspekhi*, 42(1):7–36, 1999.

[2] R. Benzi, G. Parisi, A. Sutera, and A. Vulpiani. The mechanism of stochastic resonance. *J. Phys. A*, 14:453–457, 1981.

[3] R. Benzi, G. Parisi, A. Sutera, and A. Vulpiani. Stochastic resonance in climatic changes. *Tellus*, 34:10–16, 1982.

[4] R. Benzi, G. Parisi, A. Sutera, and A. Vulpiani. A theory of stochastic resonance in climatic change. *SIAM J. Appl. Math.*, 43:563–578, 1983.

[5] A. Bovier, M. Eckhoff, V. Gayrard, and M. Klein. Metastability in reversible diffusion processes I. Sharp asymptotics for capacities and exit times. Preprint No. 767, Weierstraß-Institut für angewandte Analysis und Stochastik (WIAS), Berlin, 2002.

[6] A. Bovier, V. Gayrard, and M. Klein. Metastability in reversible diffusion processes II. Precise asymptotics for small eigenvalues. Preprint No. 768, Weierstraß-Institut für angewandte Analysis und Stochastik (WIAS), Berlin, 2002.

[7] M. I. Budyko. The effect of solar radiation variations on the climate of the earth. *Tellus*, 21:611–619, 1969.

[8] M. I. Dykman, D. G. Luchinskii, R. Mannella, P. V. E. McClintock, N. D. Stein, and N. G. Stocks. Stochastic resonance in perspective. *Nuovo Cimento D*, 17:661–683, 1995.

[9] J.-P. Eckmann and L. E. Thomas. Remarks on stochastic resonance. *J. Phys. A*, 15:261–266, 1982.

[10] A. Erdélyi. *Asymptotic expansions.* Dover Publications, Inc., New York, 1956.

[11] M. I. Freidlin. Quasi-deterministic approximation, metastability and stochastic resonance. *Physica D*, 137(3-4):333–352, 2000.

[12] M. I. Freidlin and A. D. Wentzell. *Random perturbations of dynamical systems*, volume 260 of *Grundlehren der Mathematischen Wissenschaften.* Springer, New York, NY, second edition, 1998.

[13] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni. Stochastic resonance. *Reviews of Modern Physics*, 70:223–287, January 1998.

[14] A. Ganopolski and S. Rahmstorf. Abrupt glacial climate changes due to stochastic resonance. *Physical Review Letters*, 88(3):038501+, January 2001.

[15] S. Herrmann and P. Imkeller. Barrier crossings characterize stochastic resonance. *Stochastics and Dynamics*, 2(3):413–436, 2002.

[16] P. Imkeller. Energy balance models — viewed from stochastic dynamics. In Imkeller, P. et al., editors, *Stochastic climate models. Proceedings of a workshop, Chorin, Germany, Summer 1999.*, volume 49 of *Prog. Probab.*, pages 213–240, Basel, 2001. Birkhäuser.

[17] P. Imkeller and A.H. Monahan. Conceptual stochastic climate models. *Stochastics and Dynamics*, 2(3):311–326, 2002.

[18] P. Imkeller and I. Pavlyukevich. Stochastic resonance in two-state Markov chains. *Arch. Math.*, 77(1):107–115, 2001.

[19] P. Imkeller and I. Pavlyukevich. Model reduction and stochastic resonance. *Stochastics and Dynamics*, 2(4):463–506, 2002.

[20] B. McNamara and K. Wiesenfeld. Theory of stochastic resonance. *Physical Review A (General Physics)*, 39:4854–4869, May 1989.

[21] C. Nicolis. Stochastic aspects of climatic transitions — responses to periodic forcing. *Tellus*, 34:1–9, 1982.

[22] F. W. J. Olver. *Asymptotics and special functions.* Computer Science and Applied Mathematics. Academic Press, a subsidiary of Harcourt Brace Jovanovich, Publishers, New York - London, 1974.

[23] D. Paillard. The timing of Pleistocene glaciations from a simple multiple-state climate model. *Nature*, 391:378–381, 1998.

[24] I. E. Pavlyukevich. *Stochastic Resonance.* PhD thesis, Humboldt-Universität, Berlin, 2002. Logos–Verlag, ISBN 3-89722-960-9.

[25] W. B. Sellers. A global climate model based on the energy balance of the earth-atmosphere system. *J. Appl. Meteor.*, 8:301–320, 1969.

[26] K. Wiesenfeld and F. Jaramillo. Minireview of stocastic resonance. *Chaos*, 8:539–548, September 1998.