Set theory

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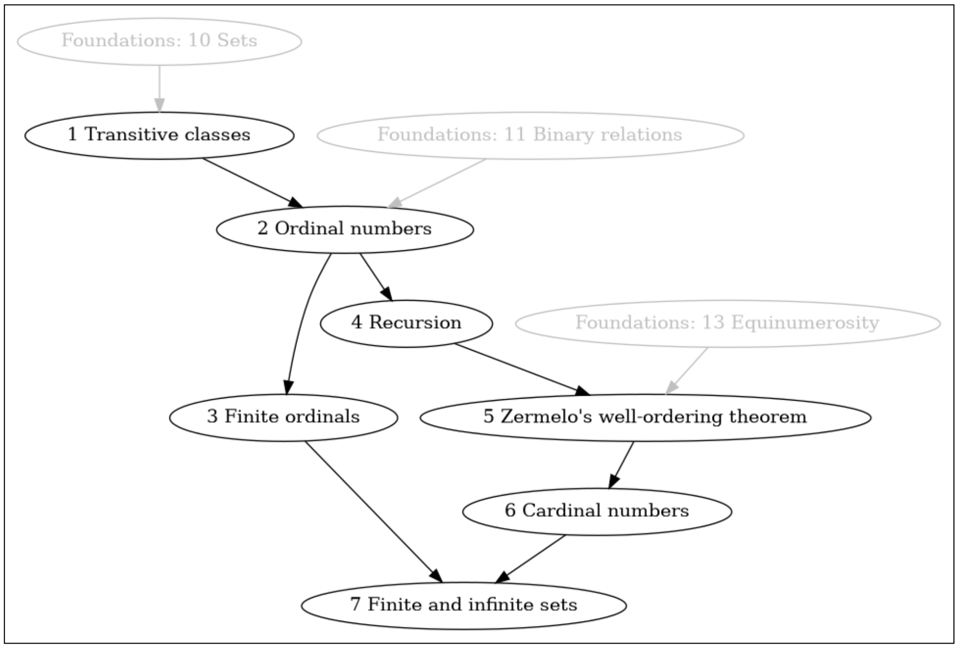
Contents



Interdependencies of the chapters

Introduction

This is a library providing basic results from undergraduate-level set theory. It introduces the notion of transitive classes (chapter 1), defines the notion of ordinal numbers (chapter 2) and as a special case of the latter introduces the set ω of finite ordinals (chapter 3). Moreover, this library provides a formalization of the ordinal recursion theorem (chapter 4) which is used to prove Zermelo's well-ordering theorem (chapter 5), on the basis of which the notion of cardinal numbers is introduced (chapter 6). Furthermore, some results about finite and infinite sets are given (chapter 7).

Usage. At the very beginning of each chapter you can find the name of its source file, e.g. `set-theory/sections/01_transitive-classes.ftl.tex` for chapter 1. This filename can be used to import the chapter via ℕaproche's `readtex` instruction to another ForTheL text, e.g.:

```
[readtex \path{set-theory/sections/01_transitive-classes.ftl.tex}]
```

Checking times. The checking times for each of the chapters may vary from computer to computer, but on mid-range hardware they are likely to be similar to those given in table below:

	Checking time	
Chapter	**without dependencies**	**with dependencies**
1	00:20 min	07:00 min
2	03:20 min	10:30 min
3	01:15 min	11:45 min
4	04:05 min	14:35 min
5	04:45 min	21:40 min
6	05:10 min	26:50 min
7	10:50 min	38:55 min

Chapter 1

Transitive classes

File: `set-theory/sections/01_transitive-classes.ftl.tex`

[readtex `foundations/sections/10_sets.ftl.tex`]

SET_THEORY_01_8167915266244608

Definition 1.1. Let A be a class. A is transitive iff every element of A is a subset of A.

SET_THEORY_01_6964770955591680

Proposition 1.2. Let X be a system of sets. Then X is transitive iff for every $x \in X$ and every $y \in x$ we have $y \in X$.

SET_THEORY_01_4219967964708864

Definition 1.3. A system of transitive sets is a system of sets X such that every member of X is a transitive set.

SET_THEORY_01_2095807333400576

Proposition 1.4. Every transitive class is a system of sets.

SET_THEORY_01_6524117649981440

Proposition 1.5. Let X be a system of sets. Then X is transitive iff $\bigcup X \subseteq X$.

Proof. Case X is transitive. Let $x \in \bigcup X$. Take a member y of X such that $x \in y$. Then y is a subset of X. Hence x is an element of X. End.

Case $\bigcup X \subseteq X$. Let $x \in X$.

Let us show that $x \subseteq X$. Let $y \in x$. Then $y \in \bigcup X$. Hence $y \in X$. End. End. □

SET_THEORY_01_620651482185728

Proposition 1.6. Let A be a transitive class. Then $\bigcup A$ is transitive.

Proof. Let $x \in \bigcup A$.

Let us show that $x \subseteq \bigcup A$. Let $y \in x$. Take a member z of A such that $x \in z$. Then $z \subseteq A$. Hence $x \in A$. Thus y is an element of some member of A. Therefore $y \in \bigcup A$. End. □

SET_THEORY_01_6726468811882496

Proposition 1.7. Let X be a system of transitive sets. Then $\bigcup X$ is transitive.

Proof. Let $x \in \bigcup X$ and $y \in x$. Take $z \in X$ such that $x \in z$. Then z is transitive. Hence $x \subseteq z$. Thus $y \in z$. Therefore $y \in \bigcup X$. □

SET_THEORY_01_4884401668227072

Proposition 1.8. Let X be a system of transitive sets. Then $X \cup \bigcup X$ is transitive.

Proof. Let $x \in X \cup \bigcup X$.

Let us show that $x \subseteq X \cup \bigcup X$. Let $u \in x$. We have $x \in X$ or $x \in \bigcup X$. If $x \in X$then $u \in \bigcup X$. If $x \in \bigcup X$ then $u \in \bigcup X$. Indeed $\bigcup X$ is transitive. Hence $u \in \bigcup X$. Thus $u \in X \cup \bigcup X$. End. □

SET_THEORY_01_1399002962591744

Proposition 1.9. Let X be a system of sets. Then X is transitive iff $X \subseteq \mathcal{P}(X)$.

Proof. Case X is transitive. Let $x \in X$. Then $x \subseteq X$. Hence $x \in \mathcal{P}(X)$. End.

Case $X \subseteq \mathcal{P}(X)$. Let $x \in X$. Then $x \in \mathcal{P}(X)$. Hence $x \subseteq X$. End. □

SET_THEORY_01_6995689103949824

Proposition 1.10. Let A be a transitive class. Then $\mathcal{P}(A)$ is transitive.

Proof. Let $x \in \mathcal{P}(A)$. Then $x \subseteq A$.

Let us show that $x \subseteq \mathcal{P}(A)$. Let $y \in x$. Then $y \in A$. Hence $y \subseteq A$. Thus $y \in \mathcal{P}(A)$. End. □

Chapter 2

Ordinal numbers

File: `set-theory/sections/02_ordinals.ftl.tex`

[readtex `foundations/sections/11_binary-relations.ftl.tex`]

[readtex `set-theory/sections/01_transitive-classes.ftl.tex`]

SET_THEORY_02_229593678086144

Definition 2.1. An ordinal number is a transitive set α such that every element of α is a transitive set.

Let an ordinal stand for an ordinal number.

SET_THEORY_02_5852994258075648

Definition 2.2. Ord is the class of all ordinals.

SET_THEORY_02_2358097091756032

Proposition 2.3. Let α be an ordinal. Then every element of α is an ordinal.

Proof. Let x be an element of α. Then x is transitive.

Let us show that every element of x is a subset of x. Let y be an element of x. Then y is a subset of x. Let z be an element of y. Every element of y is an element of x. Hence z is an element of x. End.

Thus every element of x is transitive. Therefore x is an ordinal. □

SET_THEORY_02_7202164443185152

Proposition 2.4. Let α be an ordinal and $x \subseteq \alpha$. Then $\bigcup x$ is an ordinal.

Proof. (1) $\bigcup x$ is transitive.
Proof. Let $y \in \bigcup x$ and $z \in y$. Take $w \in x$ such that $y \in w$. Then $w \in \alpha$. Hence w is transitive. Thus $z \in w$. Therefore $z \in \bigcup x$. Qed.

(2) Every element of $\bigcup x$ is transitive.
Proof. Let $y \in \bigcup x$. Let $z \in y$ and $v \in z$. Take $w \in x$ such that $y \in w$. We have $w \in \alpha$. Hence w is an ordinal. Thus y is an ordinal. Therefore y is transitive. Consequently $v \in y$. Qed. □

2.1 Zero and successors

SET_THEORY_02_8385964858671104

Definition 2.5. $0 = \emptyset$.

Let α is nonzero stand for $\alpha \neq 0$.

SET_THEORY_02_8166925802668032

Definition 2.6. Let α be an ordinal. $\operatorname{succ}(\alpha) = \alpha \cup \{\alpha\}$.

SET_THEORY_02_8483196888940544

Proposition 2.7. 0 is an ordinal.

Proof. Every element of 0 is a transitive set and every element of 0 is a subset of 0. □

SET_THEORY_02_1624410224066560

Proposition 2.8. Let α be an ordinal. Then $\operatorname{succ}(\alpha)$ is an ordinal.

Proof. (1) $\operatorname{succ}(\alpha)$ is transitive.
Proof. Let $x \in \operatorname{succ}(\alpha)$ and $y \in x$. Then $x \in \alpha$ or $x = \alpha$. Hence $y \in \alpha$. Thus

$y \in \mathrm{succ}(\alpha)$. Qed.

(2) Every element of $\mathrm{succ}(\alpha)$ is transitive.
Proof. Let $x \in \mathrm{succ}(\alpha)$. Then $x \in \alpha$ or $x = \alpha$. Hence x is transitive. Indeed α is transitive and every element of α is transitive. Qed. □

SET_THEORY_02_8651096763400192

Proposition 2.9. Let α, β be ordinals. If $\mathrm{succ}(\alpha) = \mathrm{succ}(\beta)$ then $\alpha = \beta$.

Proof. Assume $\mathrm{succ}(\alpha) = \mathrm{succ}(\beta)$.

(1) $\alpha \subseteq \beta$.
Proof. Let $\gamma \in \alpha$. Then $\gamma \in \alpha \cup \{\alpha\} = \mathrm{succ}(\alpha) = \mathrm{succ}(\beta) = \beta \cup \{\beta\}$. Hence $\gamma \in \beta$ or $\gamma = \beta$. Assume $\gamma = \beta$. Then $\beta \in \alpha$. Hence $\beta = (\beta \cup \{\beta\}) \setminus \{\gamma\} = (\alpha \cup \{\alpha\}) \setminus \{\gamma\} = (\alpha \setminus \{\gamma\}) \cup \{\alpha\}$. Therefore $\alpha \in \beta$. Consequently $\alpha \in \beta \in \alpha$. Contradiction. Qed.

(2) $\beta \subseteq \alpha$.
Proof. Let $\gamma \in \beta$. Then $\gamma \in \beta \cup \{\beta\} = \mathrm{succ}(\beta) = \mathrm{succ}(\alpha) = \alpha \cup \{\alpha\}$. Hence $\gamma \in \alpha$ or $\gamma = \alpha$. Assume $\gamma = \alpha$. Then $\alpha \in \beta$. Hence $\alpha = (\alpha \cup \{\alpha\}) \setminus \{\gamma\} = (\beta \cup \{\beta\}) \setminus \{\gamma\} = (\beta \setminus \{\gamma\}) \cup \{\beta\}$. Therefore $\beta \in \alpha$. Consequently $\beta \in \alpha \in \beta$. Contradiction. Qed. □

2.2 The standard ordering of the ordinals

SET_THEORY_02_6654252130762752

Definition 2.10. Let α, β be ordinals. α is less than β iff $\alpha \in \beta$.

Let $\alpha < \beta$ stand for α is less than β. Let $\alpha \not< \beta$ stand for not $\alpha < \beta$.

Let α is greater than β stand for $\beta < \alpha$. Let $\alpha > \beta$ stand for $\beta < \alpha$. Let $\alpha \not> \beta$ stand for not $\alpha > \beta$.

SET_THEORY_02_2639956210089984

Definition 2.11. Let α, β be ordinals. α is less than or equal to β iff $\alpha < \beta$ or $\alpha = \beta$.

Let $\alpha \leq \beta$ stand for α is less than or equal to β. Let $\alpha \nleq \beta$ stand for not $\alpha \leq \beta$.

Let α is greater than or equal to β stand for $\beta \leq \alpha$. Let $\alpha \geq \beta$ stand for $\beta \leq \alpha$. Let $\alpha \ngeq \beta$ stand for not $\alpha \geq \beta$.

SET_THEORY_02_3089369577553920

Proposition 2.12. Let α, β be ordinals. Then

$$\alpha \leq \beta \quad \text{implies} \quad \alpha \subseteq \beta.$$

Proof. Case $\alpha \leq \beta$. Then $\alpha < \beta$ or $\alpha = \beta$. Let $x \in \alpha$. If $\alpha < \beta$ then $x \in \alpha \in \beta$. Hence if $\alpha < \beta$ then $x \in \beta$. If $\alpha = \beta$ then $x \in \beta$. Thus $x \in \beta$. End. □

SET_THEORY_02_6229364135952384

Proposition 2.13. Let α be an ordinal. Then

$$\alpha \nless \alpha.$$

Proof. Assume $\alpha < \alpha$. Then $\alpha \in \alpha$. Contradiction. □

SET_THEORY_02_7098683017396224

Proposition 2.14. Let α, β, γ be ordinals. Then

$$(\alpha < \beta \text{ and } \beta < \gamma) \quad \text{implies} \quad \alpha < \gamma.$$

Proof. Assume $\alpha < \beta$ and $\beta < \gamma$. Then $\alpha \in \beta \in \gamma$. Hence $\alpha \in \gamma$. Thus $\alpha < \gamma$. □

SET_THEORY_02_1718825707896832

Proposition 2.15. Let α, β be ordinals. Then $\alpha < \beta$ or $\alpha = \beta$ or $\alpha > \beta$.

Proof. Assume the contrary. Define

$$A = \left\{ \alpha' \in \mathbf{Ord} \;\middle|\; \begin{array}{l} \text{there exists an ordinal } \beta' \text{ such that neither } \alpha' < \beta' \text{ nor } \alpha' = \beta' \\ \text{nor } \alpha' > \beta' \end{array} \right\}.$$

A is nonempty. Hence we can take a least element α' of A regarding $\in$. Define

$$B = \{\beta' \in \mathbf{Ord} \mid \text{neither } \alpha' < \beta' \text{ nor } \alpha' = \beta' \text{ nor } \alpha' > \beta'\}.$$

B is nonempty. Hence we can take a least element β' of B regarding $\in$.

Let us show that $\alpha' \subseteq \beta'$. Let $a \in \alpha'$. Then $a < \beta'$ or $a = \beta'$ or $a > \beta'$. Indeed if neither $a < \beta'$ nor $a = \beta'$ nor $a > \beta'$ then $a \in A$. If $a = \beta'$ then $\beta' < \alpha'$. If $a > \beta'$ then $\beta' < \alpha'$. Hence $a < \beta'$. Thus $a \in \beta'$. End.

Let us show that $\beta' \subseteq \alpha'$. Let $b \in \beta'$. Then $b < \alpha'$ or $b = \alpha'$ or $b > \alpha'$. If $b = \alpha'$ then $\alpha' < \beta'$. If $b > \alpha'$ then $\alpha' < \beta'$. Hence $b < \alpha'$. Thus $b \in \alpha'$. End.

Hence $\alpha' = \beta'$. Contradiction. □

SET_THEORY_02_610496856195072

Proposition 2.16. Let α, β be ordinals. Then

$$\alpha \subseteq \beta \quad \text{implies} \quad \alpha \leq \beta.$$

Proof. Assume $\alpha \subseteq \beta$.

Case $\alpha = \beta$. Trivial.

Case $\alpha \neq \beta$. Then $\alpha < \beta$ or $\alpha > \beta$. Assume $\alpha > \beta$. Then $\beta \in \alpha$. Hence $\beta \in \beta$. Contradiction. End. □

SET_THEORY_02_5689190964527104

Proposition 2.17. Let α be an ordinal. Then

$$\alpha < \mathrm{succ}(\alpha).$$

SET_THEORY_02_4064972025888768

Proposition 2.18. Let α, β be ordinals. Then

$$\beta < \mathrm{succ}(\alpha) \quad \text{implies} \quad \beta \leq \alpha.$$

Proof. Assume $\beta < \mathrm{succ}(\alpha)$. Then $\beta \in \mathrm{succ}(\alpha) = \alpha \cup \{\alpha\}$. Hence $\beta \in \alpha$ or $\beta \in \{\alpha\}$. Thus $\beta < \alpha$ or $\beta = \alpha$. Therefore $\beta \leq \alpha$. □

SET_THEORY_02_8242798790705152

Proposition 2.19. Let α be an ordinal. There exists no ordinal β such that $\alpha < \beta < \mathrm{succ}(\alpha)$.

Proof. Assume the contrary. Consider an ordinal β such that $\alpha < \beta < \mathrm{succ}(\alpha)$. Then $\beta < \alpha$ or $\beta = \alpha$. Hence $\alpha < \alpha$. Contradiction. □

2.3 Successor and limit ordinals

SET_THEORY_02_7129712109289472

Definition 2.20. A successor ordinal is an ordinal α such that $\alpha = \mathrm{succ}(\beta)$ for some ordinal β.

SET_THEORY_02_4240355610329088

Proposition 2.21. Let α be an ordinal. There exists no ordinal β such that $\alpha < \beta < \mathrm{succ}(\alpha)$.

Proof. Assume the contrary. Choose an ordinal β such that $\alpha < \beta < \mathrm{succ}(\alpha)$. Then $\alpha \in \beta \in \alpha \cup \{\alpha\}$. Hence $\beta \in \alpha$ or $\beta = \alpha$. Then $\alpha \in \alpha$. Contradiction. □

SET_THEORY_02_735071524880384

Definition 2.22. Let α be a successor ordinal. $\mathrm{pred}(\alpha)$ is the ordinal β such that $\alpha = \mathrm{succ}(\beta)$.

SET_THEORY_02_7678388934279168

Definition 2.23. A limit ordinal is an ordinal λ such that neither λ is a successor ordinal nor $\lambda = 0$.

SET_THEORY_02_4659024620421120

Proposition 2.24. Let λ be a limit ordinal and $\alpha \in \lambda$. Then λ contains $\mathrm{succ}(\alpha)$.

Proof. If $\mathrm{succ}(\alpha) \notin \lambda$ then $\alpha < \lambda < \mathrm{succ}(\alpha)$. □

SET_THEORY_03_2217148434874368

Theorem 2.25 (Burali-Forti). **Ord** is a proper class.

Proof. Assume that **Ord** is a set. **Ord** is transitive and every element of **Ord** is transitive. Hence **Ord** is an ordinal. Thus $\mathbf{Ord} \in \mathbf{Ord}$. Contradiction. □

2.4 Transfinite induction

SET_THEORY_02_4059354166722560

Definition 2.26.

$$< = \{(\alpha, \beta) \mid \alpha \text{ and } \beta \text{ are ordinals such that } \alpha < \beta\}.$$

SET_THEORY_02_4859038791630848

Proposition 2.27. $<$ is a strong wellorder on **Ord**.

Proof. For any ordinals α, β we have $(\alpha, \beta) \in <$ iff $\alpha < \beta$.

(1) $<$ is irreflexive on **Ord**. Indeed for any ordinal α we have $\alpha \not< \alpha$.

(2) $<$ is transitive on **Ord**. Indeed for any ordinals α, β, γ if $\alpha < \beta$ and $\beta < \gamma$ then $\alpha < \gamma$.

(3) $<$ is connected on **Ord**. Indeed for any distinct ordinals α, β we have $\alpha < \beta$ or $\beta < \alpha$.

Hence $<$ is a strict linear order on **Ord**.

(4) $<$ is wellfounded on **Ord**.
Proof. Let A be a nonempty subclass of **Ord**. Then we can take a least element α of A regarding $\in$. Then α is a least element of A regarding $<$. Qed.

Hence $<$ is strongly wellfounded on **Ord**. Indeed for any $\beta \in$ **Ord** we have $\beta = \{\alpha \in$ **Ord** $\mid (\alpha, \beta) \in <\}$. Thus $<$ is a strong wellorder on **Ord**. □

SET_THEORY_02_1042046129274880

Corollary 2.28. Let A be a subclass of **Ord**. If A is nonempty then A has a least element regarding $<$.

SET_THEORY_02_1991423647809536

Corollary 2.29. Let A be a subclass of **Ord**. If A is nonempty then A has a least element regarding $\in$.

SET_THEORY_03_8114657499807744

Proposition 2.30. $<$ is a strong wellorder on any ordinal.

Proof. Let α be an ordinal. For all $\beta, \gamma \in \alpha$ we have $(\beta, \gamma) \in <$ iff $\beta < \gamma$.

(1) $<$ is irreflexive on α. Indeed for all $\beta \in \alpha$ we have $\alpha \not< \alpha$.

(2) $<$ is transitive on α. Indeed for all $\beta, \gamma, \delta \in \alpha$ if $\beta < \gamma$ and $\gamma < \delta$ then $\beta < \delta$.

(3) $<$ is connected on α. Indeed for any distinct $\beta, \gamma \in \alpha$ we have $\beta < \gamma$ or $\gamma < \beta$.

Hence $<$ is a strict linear order on α.

(4) $<$ is wellfounded on α.
Proof. Let A be a nonempty subclass of α. Then we can take a least element β of A regarding $<$. Indeed A is a subclass of **Ord**. Qed.

Hence $<$ is strongly wellfounded on α. Indeed for any $\gamma \in \alpha$ we have $\gamma = \{\beta \in \mathbf{Ord} \mid (\beta, \gamma) \in <\}$. Thus $<$ is a strong wellorder on α. [unfold off] □

Note: In the proof below 11.24 refers to the Foundations library!

SET_THEORY_02_8493935460614144

Theorem 2.31. Let Φ be a class. Assume that for all ordinals α if Φ contains all ordinals less than α then Φ contains α. Then Φ contains every ordinal.

Proof. Define $B = \{x \mid x$ is a set and if $x \in \mathbf{Ord}$ then $x \in \Phi\}$.

Let us show that for all sets x if B contains every element of x that is a set then B contains x. Let x be a set. Assume that every element of x that is a set is contained in B.

Case $x \notin \mathbf{Ord}$. Trivial.

Case $x \in \mathbf{Ord}$. Then Φ contains all ordinals less than x. Hence Φ contains x. Thus $x \in B$. End. End.

[prover vampire] Hence B contains every set (by 11.24). Thus Φ contains every ordinal. □

SET_THEORY_02_7892040431960064

Theorem 2.32. Let Φ be a class. (Initial case) Assume that Φ contains 0. (Successor step) Assume that for all ordinals α if $\alpha \in \Phi$ then $\mathrm{succ}(\alpha) \in \Phi$. (Limit step) Assume that for all limit ordinals λ if every ordinals less than λ is contained in Φ then $\lambda \in \Phi$. Then Φ contains every ordinal.

Proof. Let us show that for all ordinals α if Φ contains all ordinals less than α then

Φ contains α. Let α be an ordinal. Then $\alpha = 0$ or α is a successor ordinal or α is a limit ordinal. Assume that Φ contains all ordinals less than α.

Case $\alpha = 0$. Trivial.

Case α is a successor ordinal. Take an ordinal β such that $\alpha = \mathrm{succ}(\beta)$. Then $\beta \in \Phi$. Hence $\alpha \in \Phi$ (by successor step). End.

Case α is a limit ordinal. Then $\beta \in \Phi$ for all ordinals β less than α. Hence $\alpha \in \Phi$ (by limit step). End. End.

[prover vampire] Thus Φ contains every ordinal (by 2.31). □

Chapter 3

Finite ordinals

File: `set-theory/sections/03_finite-ordinals.ftl.tex`

[readtex `set-theory/sections/02_ordinals.ftl.tex`]

SET_THEORY_03_4310076227584000

Definition 3.1.

$$\omega = \left\{ n \in \mathbf{Ord} \;\middle|\; \begin{array}{l} n \in X \text{ for every } X \subseteq \mathbf{Ord} \text{ such that } 0 \in X \text{ and for all} \\ x \in X \text{ we have } \mathrm{succ}(x) \in X \end{array} \right\}.$$

SET_THEORY_03_3576717620805632

Proposition 3.2. $0 \in \omega$.

SET_THEORY_03_8807317141192704

Proposition 3.3. Let $n \in \omega$. Then $\mathrm{succ}(n) \in \omega$.

SET_THEORY_03_344585425387520

Proposition 3.4. Let $\Phi \subseteq \omega$. Assume that $0 \in \Phi$ and for every $x \in \Phi$ we have $\mathrm{succ}(x) \in \Phi$. Then $\Phi = \omega$.

Proof. Suppose $\Phi \neq \omega$. Consider an element n of ω that is not contained in Φ. Take $\Phi' = \Phi \setminus \{n\}$.

(1) $0 \in \Phi'$. Indeed $0 \in \Phi$ and $0 \neq n$.

(2) For each $x \in \Phi'$ we have $\mathrm{succ}(x) \in \Phi'$.
Proof. Let $x \in \Phi'$. Then $\mathrm{succ}(x) \in \Phi$.

Let us show that $\mathrm{succ}(x) \neq n$. Assume $\mathrm{succ}(x) = n$. Then $x \notin \Phi$. Indeed $n \notin \Phi$ and if $x \in \Phi$ then $n = \mathrm{succ}(x) \in \Phi$. Contradiction. End.

Thus $\mathrm{succ}(x) \in \Phi'$. Qed.

Therefore every element of ω lies in Φ'. Indeed $\Phi' \subseteq \mathbf{Ord}$. Consequently $n \in \Phi'$. Contradiction. □

SET_THEORY_03_4847727433220096

Corollary 3.5. ω is a set.

Proof. Define $f(n) = \mathrm{succ}(n)$ for $n \in \omega$. Take a subset X of ω that is inductive regarding 0 and f. Indeed f is a map from ω to ω. Then we have $0 \in X$ and for each $n \in X$ we have $\mathrm{succ}(n) \in X$. Thus $X = \omega$. Therefore ω is a set. □

SET_THEORY_03_5885789275684864

Proposition 3.6. Let $n \in \omega$. Then $n = 0$ or $n = \mathrm{succ}(m)$ for some $m \in \omega$.

Proof. Assume the contrary. Consider a $k \in \omega$ such that neither $k = 0$ nor $k = \mathrm{succ}(m)$ for some $m \in \omega$. Take a class ω' such that $\omega' = \omega \setminus \{k\}$. Then ω' is a set.

(1) $0 \in \omega'$. Indeed $k \neq 0$.

(2) For all $m \in \omega'$ we have $\mathrm{succ}(m) \in \omega'$.
Proof. Let $m \in \omega'$. Then $\mathrm{succ}(m) \neq k$. Hence $\mathrm{succ}(m) \in \omega'$. Qed.

Thus every element of ω is contained in ω'. Therefore $k \in \omega'$. Contradiction. □

SET_THEORY_03_5057540872208384

Proposition 3.7. Every element of ω is an ordinal.

SET_THEORY_03_764451995254784

Proposition 3.8. ω is a limit ordinal.

Proof. ω is transitive.
Proof. Define $\Phi = \{n \in \omega \mid \text{for all } m \in n \text{ we have } m \in \omega\}$.

(1) $0 \in \Phi$.

(2) For all $n \in \Phi$ we have $\mathrm{succ}(n) \in \Phi$.
Proof. Let $n \in \Phi$. Then every element of n is contained in ω. Hence every element of $\mathrm{succ}(n)$ is contained in ω. Thus $\mathrm{succ}(n) \in \Phi$. Qed.

Therefore $\omega \subseteq \Phi$. Consequently ω is transitive. Qed.

Every element of ω is an ordinal. Hence every element of ω is transitive. Thus ω is an ordinal.

ω is a limit ordinal.
Proof. Assume the contrary. We have $\omega \neq 0$. Hence ω is a successor ordinal. Take an ordinal α such that $\mathrm{succ}(\alpha) = \omega$. Then $\alpha \in \omega$. Thus $\omega = \mathrm{succ}(\alpha) \in \omega$. Contradiction. Qed. □

SET_THEORY_03_5517271459954688

Proposition 3.9. Let λ be a limit ordinal. Then

$$\omega \leq \lambda.$$

Proof. Assume the contrary. Then $\lambda < \omega$. Consequently $\lambda \in \omega$. Hence $\lambda = 0$ or $\lambda = \mathrm{succ}(n)$ for some $n \in \omega$. Thus λ is not a limit ordinal. Contradiction. □

SET_THEORY_03_1991057988386816

Definition 3.10. $1 = \mathrm{succ}(0)$.

SET_THEORY_03_5809204518453248

Definition 3.11. $2 = \mathrm{succ}(1)$.

SET_THEORY_03_4388003120152576

Proposition 3.12. $1 = \{0\}$.

SET_THEORY_03_930896899211264

Proposition 3.13. $2 = \{0, 1\}$.

Chapter 4

Recursion

File: `set-theory/sections/04_recursion.ftl.tex`

[readtex `set-theory/sections/02_ordinals.ftl.tex`]

SET_THEORY_04_7107446027845632

Definition 4.1. Let A be a class and α be an ordinal.

$$A^{<\alpha} = \{f \mid f \text{ is a map from } \beta \text{ to } A \text{ for some ordinal } \beta \text{ less than } \alpha\}.$$

SET_THEORY_04_1955917673267200

Definition 4.2. Let A be a class.

$$A^{<\infty} = \{f \mid f \text{ is a map from } \alpha \text{ to } A \text{ for some ordinal } \alpha\}.$$

SET_THEORY_04_7841726894964736

Lemma 4.3. Let A be a class and f be a map to A such that $\mathrm{dom}(f)$ is a transitive subclass of **Ord** and $\alpha \in \mathrm{dom}(f)$. Then $f \restriction \alpha \in A^{<\infty}$.

SET_THEORY_04_5597213870784512

Definition 4.4. Let H be a map and $G : A^{<\infty} \to A$ for some class A such that H is a map to A. H is recursive regarding G iff $\mathrm{dom}(H)$ is a transitive subclass of $\mathbf{Ord}$ and for all $\alpha \in \mathrm{dom}(H)$ we have

$$H(\alpha) = G(H \restriction \alpha).$$

SET_THEORY_04_2876133366300672

Proposition 4.5. Let A be a class and G be a map from $A^{<\infty}$ to A. Let H, H' be maps to A that are recursive regarding G. Then

$$H(\alpha) = H'(\alpha)$$

for all $\alpha \in \mathrm{dom}(H) \cap \mathrm{dom}(H')$.

Proof. Define $\Phi = \{\alpha \in \mathbf{Ord} \mid \text{if } \alpha \in \mathrm{dom}(H) \cap \mathrm{dom}(H') \text{ then } H(\alpha) = H'(\alpha)\}$.

For all ordinals α if every ordinal less than α lies in Φ then $\alpha \in \Phi$.
Proof. Let $\alpha \in \mathbf{Ord}$. Assume that every $y \in \alpha$ lies in Φ.

Let us show that if $\alpha \in \mathrm{dom}(H) \cap \mathrm{dom}(H')$ then $H(\alpha) = H'(\alpha)$. Suppose $\alpha \in \mathrm{dom}(H) \cap \mathrm{dom}(H')$. Then $\alpha \subseteq \mathrm{dom}(H), \mathrm{dom}(H')$. Indeed $\mathrm{dom}(H)$ and $\mathrm{dom}(H')$ are transitive classes. Hence for all $y \in \alpha$ we have $y \in \mathrm{dom}(H) \cap \mathrm{dom}(H')$. Thus $H(y) = H'(y)$ for all $y \in \alpha$. Therefore $H \restriction \alpha = H' \restriction \alpha$. H and H' are recursive regarding G. Hence $H(\alpha) = G(H \restriction \alpha) = G(H' \restriction \alpha) = H'(\alpha)$. End.

Thus $\alpha \in \Phi$. Qed.

[prover vampire] Then Φ contains every ordinal (by theorem 2.31). Therefore we have $H(\alpha) = H'(\alpha)$ for all $\alpha \in \mathrm{dom}(H) \cap \mathrm{dom}(H')$. □

SET_THEORY_04_3600210873810944

Theorem 4.6 (Recursion theorem). Let A be a class and G be a map from $A^{<\infty}$ to A. Then there exists a map F from $\mathbf{Ord}$ to A that is recursive regarding G.

Proof. Every ordinal is contained in the domain of some map H to A such that H is recursive regarding G.
Proof. Define

$$\Phi = \left\{ \alpha \in \mathbf{Ord} \,\middle|\, \begin{array}{l}\alpha \text{ is contained in the domain of some map to } A \text{ that is recursive} \\ \text{regarding } G\end{array} \right\}.$$

Let us show that for every ordinal α if every ordinal less than α lies in Φ then $\alpha \in \Phi$. Let α be an ordinal. Assume that every ordinal less than α lies in Φ. Then for all $y \in \alpha$ there exists a map h to A such that h is recursive regarding G and $y \in \mathrm{dom}(h)$. Define $H'(y)$ = "choose a map h to A such that h is recursive regarding G and $y \in \mathrm{dom}(h)$ in $h(y)$" for $y \in \alpha$. Then H' is a map from α to A. We have $H' = H' \upharpoonright \alpha$. Define

$$H(\beta) = \begin{cases} H'(\beta) & : \beta < \alpha \\ G(H' \upharpoonright \beta) & : \beta = \alpha \end{cases}$$

for $\beta \in \mathrm{succ}(\alpha)$. Then $H \upharpoonright \beta \in A^{<\infty}$ for all $\beta \in \mathrm{dom}(H)$.

(a) $\mathrm{dom}(H)$ is a transitive subclass of **Ord**.

(b) For all $\beta \in \mathrm{dom}(H)$ we have $H(\beta) = G(H \upharpoonright \beta)$.
Proof. Let $\beta \in \mathrm{dom}(H)$. Then $\beta < \alpha$ or $\beta = \alpha$.

Case $\beta < \alpha$. Choose a map h to A such that h is recursive regarding G and $\beta \in \mathrm{dom}(h)$ and $H'(\beta) = h(\beta)$.

Let us show that for all $y \in \beta$ we have $h(y) = H(y)$. Let $y \in \beta$. Then $H(y) = H'(y)$. Choose a map h' to A such that h' is recursive regarding G and $y \in \mathrm{dom}(h')$ and $H'(y) = h'(y)$. [prover vampire] Then $h'(y) = h(y)$ (by proposition 4.5). Indeed $y \in \mathrm{dom}(h) \cap \mathrm{dom}(h')$. End.

Hence $h \upharpoonright \beta = H \upharpoonright \beta$. Thus $H(\beta) = H'(\beta) = h(\beta) = G(h \upharpoonright \beta) = G(H \upharpoonright \beta)$. End.

Case $\beta = \alpha$. We have $H \upharpoonright \alpha = H' \upharpoonright \alpha$. End. Qed.

Hence H is a map to A such that H is recursive regarding G and $\alpha \in \mathrm{dom}(H)$. Thus $\alpha \in \Phi$. End.

[prover vampire] Therefore Φ contains every ordinal (by theorem 2.31). Consequently every ordinal is contained in the domain of some map H to A such that H is recursive regarding G. Qed.

Define $F(\alpha)$ = "choose a map H to A such that H is recursive regarding G and $\alpha \in \mathrm{dom}(H)$ in $H(\alpha)$" for $\alpha \in$ **Ord**. Then F is a map from **Ord** to A.

F is recursive regarding G.
Proof. (a) $\mathrm{dom}(F)$ is a transitive subclass of **Ord**.

(b) For all $\alpha \in$ **Ord** we have $F(\alpha) = G(F \upharpoonright \alpha)$.
Proof. Let $\alpha \in$ **Ord**. Choose a map H to A such that H is recursive regarding G and $\alpha \in \mathrm{dom}(H)$ and $F(\alpha) = H(\alpha)$.

Let us show that $F(\beta) = H(\beta)$ for all $\beta \in \alpha$. Let $\beta \in \alpha$. Choose a map H' to A such that H' is recursive regarding G and $\beta \in \mathrm{dom}(H')$ and $F(\beta) = H'(\beta)$. [prover vampire] Then $H(\beta) = H'(\beta)$ (by proposition 4.5). Indeed $\beta \in \mathrm{dom}(H) \cap \mathrm{dom}(H')$. Therefore $F(\beta) = H'(\beta)$. End.

Hence $H \upharpoonright \alpha = F \upharpoonright \alpha$. Thus $F(\alpha) = H(\alpha) = G(H \upharpoonright \alpha) = G(F \upharpoonright \alpha)$. Qed. Qed. □

SET_THEORY_04_1787371569807360

Theorem 4.7. Let A be a class and G be a map from $A^{<\infty}$ to A. Let F, F' be maps from **Ord** to A that are recursive regarding G. Then $F = F'$.

Proof. F and F' are recursive regarding G. [prover vampire] Then $F(\alpha) = F'(\alpha)$ for all $\alpha \in \mathrm{dom}(F) \cap \mathrm{dom}(F')$ (by proposition 4.5). Indeed let $\alpha \in \mathrm{dom}(F) \cap \mathrm{dom}(F')$. We have $\mathrm{dom}(F) = \mathbf{Ord} = \mathrm{dom}(F')$. Hence $F(\alpha) = F'(\alpha)$ for all $\alpha \in \mathbf{Ord}$. Thus $F = F'$. □

SET_THEORY_04_8446954438656000

Theorem 4.8. Let A be a class. Let $a \in A$ and $G : \mathbf{Ord} \times A \to A$ and $H : \mathbf{Ord} \times A^{<\infty} \to A$. Then there exists a map F from **Ord** to A such that

$$F(0) = a$$

and for all ordinals α we have

$$F(\mathrm{succ}(\alpha)) = G(\alpha, F(\alpha))$$

and for all limit ordinals λ we have

$$F(\lambda) = H(\lambda, F \upharpoonright \lambda).$$

Proof. Define

$$J(f) = \begin{cases} a & : \mathrm{dom}(f) = 0 \\ G(\mathrm{pred}(\mathrm{dom}(f)), f(\mathrm{pred}(\mathrm{dom}(f)))) & : \mathrm{dom}(f) \text{ is a successor ordinal} \\ H(\mathrm{dom}(f), f) & : \mathrm{dom}(f) \text{ is a limit ordinal} \end{cases}$$

for $f \in A^{<\infty}$.

Then J is a map from $A^{<\infty}$ to A. Indeed we can show that for any $f \in A^{<\infty}$ we have $J(f) \in A$. Let $f \in A^{<\infty}$. Take $\alpha \in \mathbf{Ord}$ such that $f : \alpha \to A$. If $\alpha = 0$ then $J(f) = a \in A$. If α is a successor ordinal then $J(f) = G(\mathrm{pred}(\alpha), f(\mathrm{pred}(\alpha))) \in A$. If α is a limit ordinal then $J(f) = H(\alpha, f) \in A$. End.

Hence we can take a map F from **Ord** to A that is recursive regarding J. Then $F \upharpoonright \alpha \in A^{<\infty}$ for any ordinal α.

(1) $F(0) = a$.
Proof. $F(0) = J(F \upharpoonright 0) = a$. Qed.

(2) $F(\mathrm{succ}(\alpha)) = G(\alpha, F(\alpha))$ for all ordinals α.
Proof. Let α be an ordinal. Then $F(\mathrm{succ}(\alpha)) = J(F \upharpoonright \mathrm{succ}(\alpha)) = G(\mathrm{pred}(\mathrm{succ}(\alpha)), (F \upharpoonright \mathrm{succ}(\alpha))(\mathrm{pred}(\mathrm{succ}(\alpha)))) = G(\alpha, (F \upharpoonright \mathrm{succ}(\alpha))(\alpha)) = G(\alpha, F(\alpha))$. Qed.

(3) $F(\lambda) = H(\lambda, F \upharpoonright \lambda)$ for all limit ordinals λ.
Proof. Let λ be a limit ordinal. Then $F(\lambda) = J(F \upharpoonright \lambda) = H(\lambda, F \upharpoonright \lambda)$. Qed. □

Chapter 5

Zermelo's well-ordering theorem

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[readtex `foundations/sections/13_equinumerosity.ftl.tex`]

[readtex `set-theory/sections/04_recursion.ftl.tex`]

SET_THEORY_05_4227480655233024

Theorem 5.1 (Zermelo). Every set is equinumerous to some ordinal.

Proof. Let x be a set.

[prover vampire] Every element of $(x \cup \{x\})^{<\infty}$ is a map. Define

$$G(F) = \begin{cases} \text{choose } u \in x \setminus \text{range}(F) \text{ in } u & : x \setminus \text{range}(F) \neq \emptyset \\ x & : x \setminus \text{range}(F) = \emptyset \end{cases}$$

for $F \in (x \cup \{x\})^{<\infty}$. [prover eprover]

Then G is a map from $(x \cup \{x\})^{<\infty}$ to $x \cup \{x\}$. Indeed we can show that for any $F \in (x \cup \{x\})^{<\infty}$ we have $G(F) \in x \cup \{x\}$. Let $F \in (x \cup \{x\})^{<\infty}$. If $x \setminus \text{range}(F) \neq \emptyset$ then $G(F) \in x \setminus \text{range}(F)$. If $x \setminus \text{range}(F) = \emptyset$ then $G(F) = x$. Hence $G(F) \in x \cup \{x\}$. End. Hence we can take a map F from **Ord** to $x \cup \{x\}$ that is recursive regarding G. For any ordinal α we have $F \upharpoonright \alpha \in (x \cup \{x\})^{<\infty}$.

For any $\alpha \in \mathbf{Ord}$

$$x \setminus F[\alpha] \neq \emptyset \quad \text{implies} \quad F(\alpha) \in x \setminus F[\alpha]$$

and

$$x \setminus F[\alpha] = \emptyset \quad \text{implies} \quad F(\alpha) = x.$$

Proof. Let $\alpha \in \mathbf{Ord}$. We have $F[\alpha] = \{F(\beta) \mid \beta \in \alpha\}$. Hence $F[\alpha] = \{G(F \upharpoonright \beta) \mid \beta \in \alpha\}$. We have $\mathrm{range}(F \upharpoonright \alpha) = \{F(\beta) \mid \beta \in \alpha\}$. Thus $\mathrm{range}(F \upharpoonright \alpha) = F[\alpha]$.

Case $x \setminus F[\alpha] \neq \emptyset$. Then $x \setminus \mathrm{range}(F \upharpoonright \alpha) \neq \emptyset$. Hence $F(\alpha) = G(F \upharpoonright \alpha) \in x \setminus \mathrm{range}(F \upharpoonright \alpha) = x \setminus F[\alpha]$. End.

Case $x \setminus F[\alpha] \neq \emptyset$. Then $x \setminus \mathrm{range}(F \upharpoonright \alpha) = \emptyset$. Hence $F(\alpha) = G(F \upharpoonright \alpha) = x$. End. Qed.

(1) For any ordinals α, β such that $\alpha < \beta$ and $F(\beta) \neq x$ we have $F(\alpha), F(\beta) \in x$ and $F(\alpha) \neq F(\beta)$.
Proof. Let $\alpha, \beta \in \mathbf{Ord}$. Assume $\alpha < \beta$ and $F(\beta) \neq x$. Then $x \setminus F[\beta] \neq \emptyset$. (a) Hence $F(\beta) \in x \setminus F[\beta]$. We have $F[\alpha] \subseteq F[\beta]$. Thus $x \setminus F[\alpha] \neq \emptyset$. (b) Therefore $F(\alpha) \in x \setminus F[\alpha]$. Consequently $F(\alpha), F(\beta) \in x$ (by a, b). We have $F(\alpha) \in F[\beta]$ and $F(\beta) \notin F[\beta]$. Thus $F(\alpha) \neq F(\beta)$. Qed.

(2) There exists an ordinal α such that $F(\alpha) = x$.
Proof. Assume the contrary. Then F is a map from $\mathbf{Ord}$ to x.

Let us show that F is injective. Let $\alpha, \beta \in \mathbf{Ord}$. Assume $\alpha \neq \beta$. Then $\alpha < \beta$ or $\beta < \alpha$. Hence $F(\alpha) \neq F(\beta)$ (by 1). Indeed $F(\alpha), F(\beta) \neq x$. End.

Thus F is an injective map from some proper class to some set. Contradiction. Qed.

Define $\Phi = \{\alpha \in \mathbf{Ord} \mid F(\alpha) = x\}$. Φ is nonempty. Hence we can take a least element α of Φ regarding $\in$. Take $f = F \upharpoonright \alpha$. Then f is a map from α to x. Indeed for no $\beta \in \alpha$ we have $F(\beta) = x$. Indeed for all $\beta \in \alpha$ we have $(\beta, \alpha) \in \in$.

(3) f is surjective onto x.
Proof. $x \setminus F[\alpha] = \emptyset$. Hence $\mathrm{range}(f) = f[\alpha] = F[\alpha] = x$. Qed.

(4) f is injective.
Proof. Let $\beta, \gamma \in \alpha$. Assume $\beta \neq \gamma$. We have $f(\beta), f(\gamma) \neq x$. Hence $f(\beta) \neq f(\gamma)$ (by 1). Indeed $\beta < \gamma$ or $\gamma < \beta$. Qed.

Therefore f is a bijection between α and x. Consequently x and α are equinumerous. □

SET_THEORY_05_689384265351168

Corollary 5.2. For every set x there exists a strong wellorder on x.

Proof. Let x be a set. Choose an ordinal α that is equinumerous to x. Take a bijection f between x and α. Define $R = \{(u, v) \mid u, v \in x \text{ and } f(u) < f(v)\}$.

Let us show that R is a strong wellorder on x. $<$ is a strong wellorder on α. For all $u, v \in x$ we have $(u, v) \in R$ iff $f(u) < f(v)$.

(1) R is irreflexive on x. Indeed for all $u \in x$ we have $f(u) \nless f(u)$.

(2) R is transitive on x. Indeed for all $u, v, w \in x$ if $f(u) < f(v)$ and $f(v) < f(w)$ then $f(u) < f(w)$.

(3) R is connected on x.
Proof. Let $u, v \in x$. Assume $u \neq v$. Then $f(u) \neq f(v)$. Hence $f(u) < f(v)$ or $f(v) < f(u)$ (by proposition 2.15). Indeed $f(u), f(v)$ are ordinals. Qed.

Hence R is a strict linear order on x.

(4) R is wellfounded on x.
Proof. Let A be a nonempty subclass of x. Then we can take a least element β of $f[A]$ regarding $<$. Indeed $f[A]$ is a nonempty subclass of α. Then $f^{-1}(\beta)$ is a least element of A regarding R. Qed.

We can show that for all $v \in x$ there exists a set y such that $y = \{u \in x \mid (u, v) \in R\}$. Let $v \in x$. Define $y = \{u \in x \mid (u, v) \in R\}$. Then y is a set such that $y = \{u \in x \mid (u, v) \in R\}$. End. [prover vampire] Hence R is strongly wellfounded on x (by definition 11.18). Indeed R is a binary relation. Thus R is a strong wellorder on x. End. □

Chapter 6

Cardinal numbers

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[readtex `set-theory/sections/05_well-ordering-theorem.ftl.tex`]

SET_THEORY_06_8286266038681600

Definition 6.1. Let x be a set. The cardinality of x is the ordinal κ such that κ is equinumerous to x and every ordinal that is equinumerous to x is greater than or equal to κ.

Let $|x|$ stand for the cardinality of x.

SET_THEORY_06_6818986081648640

Definition 6.2. A cardinal number is an ordinal κ such that $\kappa = |x|$ for some set x.

Let a cardinal stand for a cardinal number.

SET_THEORY_06_2820082336006144

Proposition 6.3. Let κ be a cardinal. Then $|\kappa| = \kappa$.

Proof. κ is an ordinal that is equinumerous to κ. Hence $|\kappa| \leq \kappa$. Consider a set x

such that $\kappa = |x|$. Then $|\kappa|$ is an ordinal that is equinumerous to x. Hence $\kappa \leq |\kappa|$. Thus $|\kappa| = \kappa$. □

SET_THEORY_06_6920913721229312

Proposition 6.4. Let x, y be sets. Then x and y are equinumerous iff $|x| = |y|$.

Proof prover vampire.

Case x and y are equinumerous. Take a bijection f between x and y. Consider a bijection g between y and $|y|$. Then $g \circ f$ is a bijection between x and $|y|$ (by corollary 8.11). Hence x and $|y|$ are equinumerous. Thus $|y| \geq |x|$.

f^{-1} is a bijection between y and x. Consider a bijection h between x and $|x|$. Then $h \circ f^{-1}$ is a bijection between y and $|x|$ (by corollary 8.11). Hence y and $|x|$ are equinumerous. Thus $|x| \geq |y|$.

Therefore $|x| = |y|$. End.

Case $|x| = |y|$. Consider a bijection f between x and $|x|$ and a bijection g between $|y|$ and y. Then $g \circ f$ is a bijection between x and y. Hence x and y are equinumerous. End. □

[checktime 2]

SET_THEORY_06_5513850721927168

Proposition 6.5. Let x, y be sets and $f : x \hookrightarrow y$ and $a \subseteq x$. Then $|f[a]| = |a|$.

Proof. $f \restriction a$ is a bijection between a and $f[a]$. $f[a]$ is a set. Hence $|a| = |f[a]|$. □

[/checktime]

Proposition 6.6. Let κ be a cardinal and $x \subseteq \kappa$. Then $|x| \leq \kappa$.

Proof. Assume $|x| > \kappa$. Then $\kappa \subseteq |x|$. Take a bijection f between $|x|$ and x. Then $f \restriction \kappa$ is an injective map from κ to x. id_x is an injective map from x to κ. Hence x and κ are equinumerous (by theorem 13.5). Indeed x is a set. Thus $|x| = \kappa$. Contradiction. □

SET_THEORY_06_407116133171200

Proposition 6.7. Let x, y be sets. Then there exists an injective map from x to y iff $|x| \leq |y|$.

Proof. Case there exists an injective map from x to y. Consider an injective map f from x to y. Take a bijection g from $|x|$ to x and a bijection h from y to $|y|$. Then g is an injective map from $|x|$ to x and h is an injective map from y to $|y|$. [prover vampire] Hence $h \circ f$ is an injective map from x to $|y|$. Thus $(h \circ f) \circ g$ is an injective map from $|x|$ to $|y|$. [prover eprover] Therefore $|x| = ||x|| = |((h \circ f) \circ g)[|x|]|$. We have $((h \circ f) \circ g)[|x|] \subseteq |y|$. Hence $|x| \leq |y|$. End.

Case $|x| \leq |y|$. Take a bijection g from x to $|x|$ and a bijection h from $|y|$ to y. We have $|x| \subseteq |y|$. Hence g is an injective map from x to $|y|$. Take $f = h \circ g$. [prover vampire] Then f is an injective map from x to y. Indeed h is an injective map from $|y|$ to y. End. □

SET_THEORY_06_4944303633727488

Corollary 6.8. Let x be a set and $y \subseteq x$. Then $|y| \leq |x|$.

Proof. Define $f(v) = v$ for $v \in y$. Then f is an injective map from y to x. Hence $|y| \leq |x|$. □

SET_THEORY_06_192336220913664

Proposition 6.9. Let x, y be nonempty sets. Then there exists a surjective map from x onto y iff $|x| \geq |y|$.

Proof. Case there exists a surjective map from x onto y. Consider a surjective map f from x onto y. Define $g(v) =$ "choose $u \in x$ such that $f(u) = v$ in u" for $v \in y$. Then g is an injective map from y to x. Indeed we can show that g is injective. Let $v, v' \in y$. Assume $g(v) = g(v')$. Take $u \in x$ such that $f(u) = v$ and $g(v) = u$. Take $u' \in x$ such that $f(u') = v'$ and $g(v') = u'$. Then $v = f(u) = f(g(v)) = f(g(v')) = f(u') = v'$. End. Hence $|x| \geq |y|$. End.

Case $|x| \geq |y|$. Then we can take an injective map f from y to x. Then f^{-1} is a bijection between range(f) and y. Consider an element z of y. Define

$$g(u) = \begin{cases} f^{-1}(u) & : u \in \text{range}(f) \\ z & : u \notin \text{range}(f) \end{cases}$$

for $u \in x$. Then g is a surjective map from x onto y. Indeed we can show that every element of y is a value of g. Let $v \in y$. Then $f(v) \in \text{range}(f)$. Hence

$g(f(v)) = f^{-1}(f(v)) = v$. End. End. □

[checktime 2]

SET_THEORY_06_8113916590686208

Proposition 6.10. Let x, y be sets and $f : x \to y$ and $a \subseteq x$. Then $|f[a]| \leq |a|$.

Proof. Case a is empty. Obvious.

Case a is nonempty. $f \restriction a$ is a surjective map from a onto $f[a]$ and $f[a]$ is nonempty. Hence $|f[a]| \leq |a|$ (by proposition 6.9). Indeed a and $f[a]$ are sets. End. □

[/checktime]

SET_THEORY_06_5843717288099840

Proposition 6.11. Let x, y be nonempty sets. $|x| < |y|$ iff there exists an injective map from x to y and there exists no surjective map from x onto y.

Proof. There exists an injective map from x to y and there exists no surjective map from x onto y iff $|x| \leq |y|$ and $|x| \not\geq |y|$ (by proposition 6.7, proposition 6.9). $|x| \leq |y|$ and $|x| \not\geq |y|$ iff $|x| \leq |y|$ and $|x| \neq |y|$. $|x| \leq |y|$ and $|x| \neq |y|$ iff $|x| < |y|$. □

SET_THEORY_06_8300194126888960

Proposition 6.12. Let x, y be sets and $f : x \to y$ and $b \subseteq \mathrm{range}(f)$. Then $|f^*(b)| \geq |b|$.

Proof. Case b is empty. Obvious.

Case b is nonempty. $f \restriction f^*(b)$ is a surjective map from $f^*(b)$ onto b. Hence $|f^*(b)| \geq |b|$ (by proposition 6.9). Indeed b and $f^*(b)$ are nonempty sets. End. □

SET_THEORY_06_2993566311776256

Proposition 6.13. Let x, y be sets and $f : x \hookrightarrow y$ and $b \subseteq \mathrm{range}(f)$. Then $|f^*(b)| = |b|$.

Proof. $f \restriction f^*(b)$ is a bijection between $f^*(b)$ and b. Indeed $b = f[f^*(b)] = (f \restriction f^*(b))[f^*(b)] = \mathrm{range}(f \restriction f^*(b))$. Hence $|f^*(b)| = |b|$. □

SET_THEORY_06_7746592696172544

Proposition 6.14. Let x, y be sets such that $|y| < |x|$. Then $x \setminus y$ is nonempty.

Proof. Assume the contrary. Then $x \subseteq y$. Hence $|x| \leq |y|$. Contradiction. ☐

SET_THEORY_06_914271456198656

Theorem 6.15 (Cantor). Let x be a set. Then

$$|x| < |\mathcal{P}(x)|.$$

Proof. Let us show that there exists no surjective map from x onto $\mathcal{P}(x)$. Assume the contrary. Take a surjective map f from x onto $\mathcal{P}(x)$. Define $C = \{u \in x \mid u \notin f(u)\}$. Then $C \in \mathcal{P}(x)$. Hence we can take a $u \in x$ such that $f(u) = C$. Then $u \in C$ iff $u \in f(u)$ iff $u \notin C$. Contradiction. End.

Thus $|x| \ngeq |\mathcal{P}(x)|$. Therefore $|x| < |\mathcal{P}(x)|$. ☐

SET_THEORY_06_8562942165385216

Theorem 6.16. For every ordinal α there exists a cardinal greater than α.

Proof. Let α be an ordinal. Take $\kappa = |\mathcal{P}(\alpha)|$. Then $\kappa > |\alpha|$.

Let us show that $\kappa > \alpha$. Assume the contrary. Then $|\mathcal{P}(\alpha)| = \kappa \leq \alpha$. Hence $\kappa = |\mathcal{P}(\alpha)| = ||\mathcal{P}(\alpha)|| \leq |\alpha|$. Contradiction. End. ☐

Chapter 7

Finite and infinite sets

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[readtex `set-theory/sections/06_cardinals.ftl.tex`]

SET_THEORY_07_6139396896063488

Proposition 7.1. $|\emptyset| = 0$.

SET_THEORY_07_836893598023680

Proposition 7.2. Let a be an object. Then $|\{a\}| = 1$.

Proof. Define $f(x) = 0$ for $x \in \{a\}$. Then f is a map from $\{a\}$ to 1. f is injective and surjective onto 1. Hence f is a bijection between $\{a\}$ and 1. Consequently $\{a\}$ and 1 are equinumerous. Thus $|\{a\}| = |1| = 1$. □

SET_THEORY_07_5465279026954240

Proposition 7.3. Let a, b be distinct objects. Then $|\{a, b\}| = 2$.

Proof. Define

$$f(x) = \begin{cases} 0 & x = a \\ 1 & x = b \end{cases}$$

for $x \in \{a, b\}$. Then f is a map from $\{a, b\}$ to 2. f is injective and surjective onto 2. Hence f is a bijection between $\{a, b\}$ and 2. Consequently $\{a, b\}$ and 2 are equinumerous. Thus $|\{a, b\}| = |2| = 2$. □

SET_THEORY_07_2948332552978432

Theorem 7.4. Let $n \in \omega$. Then $|n| = n$.

Proof. Define $\Phi = \{n' \in \omega \mid |n'| = n'\}$.

(1) $0 \in \Phi$. Indeed $|0| = |\emptyset| = 0$.

(2) For all $n' \in \Phi$ we have $\mathrm{succ}(n') \in \Phi$.
Proof. Let $n' \in \Phi$. Then $|n'| = n'$. We have $|\mathrm{succ}(n')| \leq \mathrm{succ}(n')$.

Let us show that $\mathrm{succ}(n') \leq |\mathrm{succ}(n')|$. Assume the contrary. Then $|\mathrm{succ}(n')| < \mathrm{succ}(n')$. Take a bijection f between $|\mathrm{succ}(n')|$ and $\mathrm{succ}(n')$. $|\mathrm{succ}(n')|$ is nonzero. Hence we can take a $m \in \omega$ such that $|\mathrm{succ}(n')| = \mathrm{succ}(m)$. Then $f^{-1}(n') \leq m$.

We can show that $f^{-1}(n') < m$. Assume the contrary. Then $f^{-1}(n') = m$. $f \restriction m$ is a bijection between m and $f[m]$ (by proposition 8.13). Indeed f is an injective map from $|\mathrm{succ}(n')|$ to $\mathrm{succ}(n')$ and $m \subseteq |\mathrm{succ}(n')|$. We have $f[m] \subseteq n'$ and $n' \subseteq f[m]$. Hence $f[m] = n'$. Thus $f \restriction m$ is a bijection between m and n'. Therefore $n' = |n'| \leq m < |\mathrm{succ}(n')| < \mathrm{succ}(n')$. Consequently $m = n'$. Then we have $\mathrm{succ}(n') = |\mathrm{succ}(n')| < \mathrm{succ}(n')$. Contradiction. End.

Define

$$g(i) = \begin{cases} f(i) & : i \neq f^{-1}(n') \\ f(m) & : i = f^{-1}(n') \end{cases}$$

for $i \in m$.

g is a map from m to n'. Indeed we can show that $g(i) \in n'$ for each $i \in m$.
Proof. Let $i \in m$.

Case $i \neq f^{-1}(n')$. Then $g(i) = f(i) \in \mathrm{succ}(n')$. If $g(i) = n'$ then $f(i) = n' = f(f^{-1}(n'))$. Hence if $g(i) = n'$ then $i = f^{-1}(n')$. Thus $g(i) \neq n'$. Therefore $g(i) \in n'$. End.

Case $i = f^{-1}(n')$. Then $g(i) = f(m) \neq f(f^{-1}(n')) = n'$. Hence $g(i) \in n'$. End. Qed.

g is surjective onto n'. Indeed we can show that for all $k \in n'$ there exists a $l \in m$ such that $k = g(l)$.
Proof. Let $k \in n'$. Then $f^{-1}(k) \neq f^{-1}(n')$.

Case $f^{-1}(k) = m$. Then $k = f(f^{-1}(k)) = f(m) = g(f^{-1}(n'))$. End.

Case $f^{-1}(k) \neq m$. Then $f^{-1}(k) \in m$. Indeed $f^{-1}(k) \in |\mathrm{succ}(n')| = \mathrm{succ}(m) = m \cup \{m\}$. Hence $k = f(f^{-1}(k)) = g(f^{-1}(k))$. End. Qed.

g is injective. Indeed we can show that for all $i, j \in m$ if $i \neq j$ then $g(i) \neq g(j)$.
Proof. Let $i, j \in m$. Assume $i \neq j$.

Case $i, j \neq f^{-1}(n')$. Then $g(i) = f(i) \neq f(j) = g(j)$. End.

Case $i = f^{-1}(n')$. Then $j \neq f^{-1}(n')$. Hence $g(i) = g(f^{-1}(n')) = f(m) \neq f(j) = g(j)$. Indeed $m \neq j$. End.

Case $j = f^{-1}(n')$. Then $i \neq f^{-1}(n')$. Hence $g(i) = f(i) \neq f(m) = g(f^{-1}(n')) = g(j)$. Indeed $i \neq m$. End. Qed. End. End.

Thus $\omega \subseteq \Phi$ (by proposition 3.4). Consequently $n \in \Phi$. Therefore $|n| = n$. □

SET_THEORY_07_7061392098066432

Corollary 7.5. Every element of ω is a cardinal.

SET_THEORY_07_4952029518626816

Proposition 7.6. $|\omega| = \omega$.

Proof. We have $|\omega| \leq \omega$.

Let us show that $|\omega|$ is not less than ω. Assume the contrary. Then $|\omega| \in \omega$. Take $n = |\omega|$ and a bijection f between n and ω.

Define
$$g(k) = \begin{cases} \mathrm{succ}(f(k)) & : k < n \\ 0 & : k = n \end{cases}$$
for $k \in \mathrm{succ}(n)$. Then g is a map from $\mathrm{succ}(n)$ to ω.

g is injective. Indeed we can show that for all $k, k' \in \mathrm{succ}(n)$ if $k \neq k'$ then $g(k) \neq g(k')$.
Proof. Let $k, k' \in \mathrm{succ}(n)$. Assume $k \neq k'$.

Case $k, k' < n$. Then $f(k) \neq f(k')$. Hence $\mathrm{succ}(f(k)) \neq \mathrm{succ}(f(k'))$. Thus $g(k) \neq g(k')$. End.

Case $k < n$ and $k' = n$. We have $\mathrm{succ}(f(k)) \neq 0$. Hence $g(k) \neq g(k')$. End.

Case $k = n$ and $k' < n$. We have $\mathrm{succ}(f(k')) \neq 0$. Hence $g(k) \neq g(k')$. End. Qed.

g is surjective onto ω. Indeed we can show that for any $m \in \omega$ there exists a $k \in \mathrm{succ}(n)$ such that $m = g(k)$.
Proof. Let $m \in \omega$. Then $f^{-1}(m) \in n$.

Case $m = 0$. Then $m = g(n)$. End.

Case $m \neq 0$. Take $m' \in \omega$ such that $m = \mathrm{succ}(m')$. Then $m = \mathrm{succ}(m') =$

$\operatorname{succ}(f(f^{-1}(m'))) = g(f^{-1}(m'))$. Indeed $f^{-1}(m') < n$. End. End.

Hence g is a bijection between $\operatorname{succ}(n)$ and ω. Then we have $|n| = |\operatorname{succ}(n)|$. Thus $n = \operatorname{succ}(n)$. End. □

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Corollary 7.7. ω is a cardinal.

SET_THEORY_07_5346658235711488

Definition 7.8. Let x be a set. x is finite iff $|x| < \omega$.

SET_THEORY_07_8295412068777984

Definition 7.9. Let x be a set. x is infinite iff x is not finite.

SET_THEORY_07_8808604616359936

Definition 7.10. Let x be a set. x is countable iff $|x| \leq \omega$.

SET_THEORY_07_2935263915409408

Definition 7.11. Let x be a set. x is uncountable iff x is not countable.

SET_THEORY_07_5679866426949632

Definition 7.12. Let x be a set. x is countably infinite iff $|x| = \omega$.

SET_THEORY_07_3806229474312192

Proposition 7.13. Let x be a set. Then x is finite iff $|x| = n$ for some $n \in \omega$.

SET_THEORY_07_3174577070931968

Proposition 7.14. Let x be a set. Then x is infinite iff $|x| \geq \omega$.

Proof. $|x| \geq \omega$ iff $|x| \not< \omega$. □

SET_THEORY_07_4281623468048384

Proposition 7.15. Let x be a set. Then x is uncountable iff $|x| > \omega$.

SET_THEORY_07_4231078585827328

Definition 7.16. **Card** is the collection of all infinite cardinals.

SET_THEORY_07_4285360123150336

Proposition 7.17. **Card** is a proper class.

Proof. Suppose that **Card** is a set. Then $\bigcup \mathbf{Card}$ is a set.

Let us show that $\bigcup \mathbf{Card}$ contains every ordinal. Let α be an ordinal. Choose an infinite ordinal β such that $\beta \geq \alpha$. Choose a cardinal κ greater than β. Then $\alpha \in \kappa \in \mathbf{Card}$. Hence $\alpha \in \bigcup \mathbf{Card}$. End.

Therefore $\mathbf{Ord} \subseteq \bigcup \mathbf{Card}$. Thus **Ord** is a set. Contradiction. □

SET_THEORY_07_8189062544359424

Proposition 7.18. Let α be an infinite ordinal. Then $|\operatorname{succ}(\alpha)| = |\alpha|$.

Proof. For any $\beta \in \operatorname{succ}(\alpha)$ we have $\beta < \omega$ or $\omega \leq \beta < \alpha$ or $\beta = \alpha$. Define

$$f(\beta) = \begin{cases} \operatorname{succ}(\beta) & : \beta < \omega \\ \beta & : \omega \leq \beta < \alpha \\ 0 & : \beta = \alpha \end{cases}$$

for $\beta \in \operatorname{succ}(\alpha)$.

Then f is a map from $\operatorname{succ}(\alpha)$ to α. Indeed we can show that $f(\beta) \in \alpha$ for all $\beta \in \operatorname{succ}(\alpha)$.
Proof. Let $\beta \in \operatorname{succ}(\alpha)$.

Case $\beta < \omega$. Then $f(\beta) = \operatorname{succ}(\beta) < \omega \leq \alpha$. End.

Case $\omega \leq \beta < \alpha$. Then $f(\beta) = \beta < \alpha$. End.

Case $\beta = \alpha$. Then $f(\beta) = 0 < \alpha$. End. Qed.

f is surjective onto α. Indeed we can show that for any $\beta \in \alpha$ there exists a $\gamma \in \mathrm{succ}(\alpha)$ such that $\beta = f(\gamma)$.
Proof. Let $\beta \in \alpha$. Then $\beta = 0$ or $0 < \beta < \omega$ or $\beta \geq \omega$.

Case $\beta = 0$. Then $\beta = f(\alpha)$. End.

Case $0 < \beta < \omega$. Take an ordinal β' such that $\beta = \mathrm{succ}(\beta')$. Then $\beta' < \omega$. Hence $\beta = f(\beta')$. End.

Case $\beta \geq \omega$. Then $\beta = f(\beta)$. End. Qed.

f is injective. Indeed we can show that for all $\beta, \gamma \in \mathrm{succ}(\alpha)$ if $\beta \neq \gamma$ then $f(\beta) \neq f(\gamma)$.
Proof. Let $\beta, \gamma \in \mathrm{succ}(\alpha)$. Assume $\beta \neq \gamma$.

Case $\beta < \omega$. If $\gamma = \alpha$ then $f(\beta) = \mathrm{succ}(\beta) \neq 0 = f(\gamma)$. If $\omega \leq \gamma < \alpha$ then $f(\beta) = \mathrm{succ}(\beta) < \omega \leq \gamma = f(\gamma)$. End.

Case $\omega \leq \beta < \alpha$. If $\gamma = \alpha$ then $f(\beta) = \beta \geq \omega > 0 = f(\gamma)$. If $\gamma < \omega$ then $f(\beta) = \beta \geq \omega > \mathrm{succ}(\gamma) = f(\gamma)$. End.

Case $\beta = \alpha$. If $\gamma < \omega$ then $f(\beta) = 0 \neq \mathrm{succ}(\gamma) = f(\gamma)$. If $\omega \leq \gamma < \alpha$ then $f(\beta) = 0 < \omega \leq \gamma = f(\gamma)$. End. Qed.

Hence f is a bijection between $\mathrm{succ}(\alpha)$ and α. Therefore $\mathrm{succ}(\alpha)$ and α are equinumerous. Consequently $|\mathrm{succ}(\alpha)| = |\alpha|$. ☐

SET_THEORY_07_8700732632989696

Proposition 7.19. Every infinite cardinal is a limit ordinal.

Proof. Let κ be an infinite cardinal. Suppose that κ is not a limit ordinal. $\kappa \neq 0$. Hence κ is a successor ordinal. Thus we can take an ordinal α such that $\kappa = \mathrm{succ}(\alpha)$. We have $\alpha > \kappa \geq \omega$. Hence $|\mathrm{succ}(\alpha)| = |\alpha|$. Thus $\alpha < |\kappa|$ and κ is equinumerous to κ. Contradiction. ☐