Chapter 1

Finite and infinite sets

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[readtex `set-theory/sections/03_finite-ordinals.ftl.tex`]

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Proposition 1.1. $|\emptyset| = 0$.

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Proposition 1.2. Let a be an object. Then $|\{a\}| = 1$.

Proof. Define $f(x) = 0$ for $x \in \{a\}$. Then f is a map from $\{a\}$ to 1. f is injective and surjective onto 1. Hence f is a bijection between $\{a\}$ and 1. Consequently $\{a\}$ and 1 are equinumerous. Thus $|\{a\}| = |1| = 1$. □

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Proposition 1.3. Let a, b be distinct objects. Then $|\{a, b\}| = 2$.

Proof. Define

$$f(x) = \begin{cases} 0 & x = a \\ 1 & x = b \end{cases}$$

for $x \in \{a, b\}$. Then f is a map from $\{a, b\}$ to 2. f is injective and surjective onto 2. Hence f is a bijection between $\{a, b\}$ and 2. Consequently $\{a, b\}$ and 2 are equinumerous. Thus $|\{a, b\}| = |2| = 2$. □

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Theorem 1.4. Let $n \in \omega$. Then $|n| = n$.

Proof. Define $\Phi = \{n' \in \omega \mid |n'| = n'\}$.

(1) $0 \in \Phi$. Indeed $|0| = |\emptyset| = 0$.

(2) For all $n' \in \Phi$ we have $\mathrm{succ}(n') \in \Phi$.
Proof. Let $n' \in \Phi$. Then $|n'| = n'$. We have $|\mathrm{succ}(n')| \leq \mathrm{succ}(n')$.

Let us show that $\mathrm{succ}(n') \leq |\mathrm{succ}(n')|$. Assume the contrary. Then $|\mathrm{succ}(n')| < \mathrm{succ}(n')$. Take a bijection f between $|\mathrm{succ}(n')|$ and $\mathrm{succ}(n')$. $|\mathrm{succ}(n')|$ is nonzero. Hence we can take a $m \in \omega$ such that $|\mathrm{succ}(n')| = \mathrm{succ}(m)$. Then $f^{-1}(n') \leq m$.

We can show that $f^{-1}(n') < m$. Assume the contrary. Then $f^{-1}(n') = m$. $f \restriction m$ is a bijection between m and $f[m]$ (by proposition 8.13). Indeed f is an injective map from $|\mathrm{succ}(n')|$ to $\mathrm{succ}(n')$ and $m \subseteq |\mathrm{succ}(n')|$. We have $f[m] \subseteq n'$ and $n' \subseteq f[m]$. Hence $f[m] = n'$. Thus $f \restriction m$ is a bijection between m and n'. Therefore $n' = |n'| \leq m < |\mathrm{succ}(n')| < \mathrm{succ}(n')$. Consequently $m = n'$. Then we have $\mathrm{succ}(n') = |\mathrm{succ}(n')| < \mathrm{succ}(n')$. Contradiction. End.

Define

$$g(i) = \begin{cases} f(i) & : i \neq f^{-1}(n') \\ f(m) & : i = f^{-1}(n') \end{cases}$$

for $i \in m$.

g is a map from m to n'. Indeed we can show that $g(i) \in n'$ for each $i \in m$.
Proof. Let $i \in m$.

Case $i \neq f^{-1}(n')$. Then $g(i) = f(i) \in \mathrm{succ}(n')$. If $g(i) = n'$ then $f(i) = n' = f(f^{-1}(n'))$. Hence if $g(i) = n'$ then $i = f^{-1}(n')$. Thus $g(i) \neq n'$. Therefore $g(i) \in n'$. End.

Case $i = f^{-1}(n')$. Then $g(i) = f(m) \neq f(f^{-1}(n')) = n'$. Hence $g(i) \in n'$. End. Qed.

g is surjective onto n'. Indeed we can show that for all $k \in n'$ there exists a $l \in m$ such that $k = g(l)$.
Proof. Let $k \in n'$. Then $f^{-1}(k) \neq f^{-1}(n')$.

Case $f^{-1}(k) = m$. Then $k = f(f^{-1}(k)) = f(m) = g(f^{-1}(n'))$. End.

Case $f^{-1}(k) \neq m$. Then $f^{-1}(k) \in m$. Indeed $f^{-1}(k) \in |\mathrm{succ}(n')| = \mathrm{succ}(m) = m \cup \{m\}$. Hence $k = f(f^{-1}(k)) = g(f^{-1}(k))$. End. Qed.

g is injective. Indeed we can show that for all $i, j \in m$ if $i \neq j$ then $g(i) \neq g(j)$.
Proof. Let $i, j \in m$. Assume $i \neq j$.

Case $i, j \neq f^{-1}(n')$. Then $g(i) = f(i) \neq f(j) = g(j)$. End.

Case $i = f^{-1}(n')$. Then $j \neq f^{-1}(n')$. Hence $g(i) = g(f^{-1}(n')) = f(m) \neq f(j) = g(j)$. Indeed $m \neq j$. End.

Case $j = f^{-1}(n')$. Then $i \neq f^{-1}(n')$. Hence $g(i) = f(i) \neq f(m) = g(f^{-1}(n')) = g(j)$. Indeed $i \neq m$. End. Qed. End. End.

Thus $\omega \subseteq \Phi$ (by **??**). Consequently $n \in \Phi$. Therefore $|n| = n$. □

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Corollary 1.5. Every element of ω is a cardinal.

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Proposition 1.6. $|\omega| = \omega$.

Proof. We have $|\omega| \leq \omega$.

Let us show that $|\omega|$ is not less than ω. Assume the contrary. Then $|\omega| \in \omega$. Take $n = |\omega|$ and a bijection f between n and ω.

Define

$$g(k) = \begin{cases} \mathrm{succ}(f(k)) & : k < n \\ 0 & : k = n \end{cases}$$

for $k \in \mathrm{succ}(n)$. Then g is a map from $\mathrm{succ}(n)$ to ω.

g is injective. Indeed we can show that for all $k, k' \in \mathrm{succ}(n)$ if $k \neq k'$ then $g(k) \neq g(k')$.
Proof. Let $k, k' \in \mathrm{succ}(n)$. Assume $k \neq k'$.

Case $k, k' < n$. Then $f(k) \neq f(k')$. Hence $\mathrm{succ}(f(k)) \neq \mathrm{succ}(f(k'))$. Thus $g(k) \neq g(k')$. End.

Case $k < n$ and $k' = n$. We have $\mathrm{succ}(f(k)) \neq 0$. Hence $g(k) \neq g(k')$. End.

Case $k = n$ and $k' < n$. We have $\mathrm{succ}(f(k')) \neq 0$. Hence $g(k) \neq g(k')$. End. Qed.

g is surjective onto ω. Indeed we can show that for any $m \in \omega$ there exists a $k \in \mathrm{succ}(n)$ such that $m = g(k)$.
Proof. Let $m \in \omega$. Then $f^{-1}(m) \in n$.

Case $m = 0$. Then $m = g(n)$. End.

Case $m \neq 0$. Take $m' \in \omega$ such that $m = \mathrm{succ}(m')$. Then $m = \mathrm{succ}(m') =$

$\mathrm{succ}(f(f^{-1}(m'))) = g(f^{-1}(m'))$. Indeed $f^{-1}(m') < n$. End. End.

Hence g is a bijection between $\mathrm{succ}(n)$ and ω. Then we have $|n| = |\mathrm{succ}(n)|$. Thus $n = \mathrm{succ}(n)$. End. □

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Corollary 1.7. ω is a cardinal.

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Definition 1.8. Let x be a set. x is finite iff $|x| < \omega$.

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Definition 1.9. Let x be a set. x is infinite iff x is not finite.

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Definition 1.10. Let x be a set. x is countable iff $|x| \leq \omega$.

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Definition 1.11. Let x be a set. x is uncountable iff x is not countable.

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Definition 1.12. Let x be a set. x is countably infinite iff $|x| = \omega$.

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Proposition 1.13. Let x be a set. Then x is finite iff $|x| = n$ for some $n \in \omega$.

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Proposition 1.14. Let x be a set. Then x is infinite iff $|x| \geq \omega$.

Proof. $|x| \geq \omega$ iff $|x| \not< \omega$. □

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Proposition 1.15. Let x be a set. Then x is uncountable iff $|x| > \omega$.

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Definition 1.16. Card is the collection of all infinite cardinals.

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Proposition 1.17. Card is a proper class.

Proof. Suppose that **Card** is a set. Then $\bigcup \mathbf{Card}$ is a set.

Let us show that $\bigcup \mathbf{Card}$ contains every ordinal. Let α be an ordinal. Choose an infinite ordinal β such that $\beta \geq \alpha$. Choose a cardinal κ greater than β. Then $\alpha \in \kappa \in \mathbf{Card}$. Hence $\alpha \in \bigcup \mathbf{Card}$. End.

Therefore $\mathbf{Ord} \subseteq \bigcup \mathbf{Card}$. Thus **Ord** is a set. Contradiction. □

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Proposition 1.18. Let α be an infinite ordinal. Then $|\operatorname{succ}(\alpha)| = |\alpha|$.

Proof. For any $\beta \in \operatorname{succ}(\alpha)$ we have $\beta < \omega$ or $\omega \leq \beta < \alpha$ or $\beta = \alpha$. Define

$$f(\beta) = \begin{cases} \operatorname{succ}(\beta) & : \beta < \omega \\ \beta & : \omega \leq \beta < \alpha \\ 0 & : \beta = \alpha \end{cases}$$

for $\beta \in \operatorname{succ}(\alpha)$.

Then f is a map from $\operatorname{succ}(\alpha)$ to α. Indeed we can show that $f(\beta) \in \alpha$ for all $\beta \in \operatorname{succ}(\alpha)$.
Proof. Let $\beta \in \operatorname{succ}(\alpha)$.

Case $\beta < \omega$. Then $f(\beta) = \operatorname{succ}(\beta) < \omega \leq \alpha$. End.

Case $\omega \leq \beta < \alpha$. Then $f(\beta) = \beta < \alpha$. End.

Case $\beta = \alpha$. Then $f(\beta) = 0 < \alpha$. End. Qed.

f is surjective onto α. Indeed we can show that for any $\beta \in \alpha$ there exists a $\gamma \in \mathrm{succ}(\alpha)$ such that $\beta = f(\gamma)$.
Proof. Let $\beta \in \alpha$. Then $\beta = 0$ or $0 < \beta < \omega$ or $\beta \geq \omega$.

Case $\beta = 0$. Then $\beta = f(\alpha)$. End.

Case $0 < \beta < \omega$. Take an ordinal β' such that $\beta = \mathrm{succ}(\beta')$. Then $\beta' < \omega$. Hence $\beta = f(\beta')$. End.

Case $\beta \geq \omega$. Then $\beta = f(\beta)$. End. Qed.

f is injective. Indeed we can show that for all $\beta, \gamma \in \mathrm{succ}(\alpha)$ if $\beta \neq \gamma$ then $f(\beta) \neq f(\gamma)$.
Proof. Let $\beta, \gamma \in \mathrm{succ}(\alpha)$. Assume $\beta \neq \gamma$.

Case $\beta < \omega$. If $\gamma = \alpha$ then $f(\beta) = \mathrm{succ}(\beta) \neq 0 = f(\gamma)$. If $\omega \leq \gamma < \alpha$ then $f(\beta) = \mathrm{succ}(\beta) < \omega \leq \gamma = f(\gamma)$. End.

Case $\omega \leq \beta < \alpha$. If $\gamma = \alpha$ then $f(\beta) = \beta \geq \omega > 0 = f(\gamma)$. If $\gamma < \omega$ then $f(\beta) = \beta \geq \omega > \mathrm{succ}(\gamma) = f(\gamma)$. End.

Case $\beta = \alpha$. If $\gamma < \omega$ then $f(\beta) = 0 \neq \mathrm{succ}(\gamma) = f(\gamma)$. If $\omega \leq \gamma < \alpha$ then $f(\beta) = 0 < \omega \leq \gamma = f(\gamma)$. End. Qed.

Hence f is a bijection between $\mathrm{succ}(\alpha)$ and α. Therefore $\mathrm{succ}(\alpha)$ and α are equinumerous. Consequently $|\,\mathrm{succ}(\alpha)| = |\alpha|$. □

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Proposition 1.19. Every infinite cardinal is a limit ordinal.

Proof. Let κ be an infinite cardinal. Suppose that κ is not a limit ordinal. $\kappa \neq 0$. Hence κ is a successor ordinal. Thus we can take an ordinal α such that $\kappa = \mathrm{succ}(\alpha)$. We have $\alpha > \kappa \geq \omega$. Hence $|\,\mathrm{succ}(\alpha)| = |\alpha|$. Thus $\alpha < |\kappa|$ and κ is equinumerous to κ. Contradiction. □