Chapter 1

Cardinal numbers

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[readtex `set-theory/sections/05_well-ordering-theorem.ftl.tex`]

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Definition 1.1. Let x be a set. The cardinality of x is the ordinal κ such that κ is equinumerous to x and every ordinal that is equinumerous to x is greater than or equal to κ.

Let $|x|$ stand for the cardinality of x.

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Definition 1.2. A cardinal number is an ordinal κ such that $\kappa = |x|$ for some set x.

Let a cardinal stand for a cardinal number.

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Proposition 1.3. Let κ be a cardinal. Then $|\kappa| = \kappa$.

Proof. κ is an ordinal that is equinumerous to κ. Hence $|\kappa| \leq \kappa$. Consider a set x

such that $\kappa = |x|$. Then $|\kappa|$ is an ordinal that is equinumerous to x. Hence $\kappa \leq |\kappa|$. Thus $|\kappa| = \kappa$. ☐

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Proposition 1.4. Let x, y be sets. Then x and y are equinumerous iff $|x| = |y|$.

Proof prover vampire.

Case x and y are equinumerous. Take a bijection f between x and y. Consider a bijection g between y and $|y|$. Then $g \circ f$ is a bijection between x and $|y|$ (by corollary 8.11). Hence x and $|y|$ are equinumerous. Thus $|y| \geq |x|$.

f^{-1} is a bijection between y and x. Consider a bijection h between x and $|x|$. Then $h \circ f^{-1}$ is a bijection between y and $|x|$ (by corollary 8.11). Hence y and $|x|$ are equinumerous. Thus $|x| \geq |y|$.

Therefore $|x| = |y|$. End.

Case $|x| = |y|$. Consider a bijection f between x and $|x|$ and a bijection g between $|y|$ and y. Then $g \circ f$ is a bijection between x and y. Hence x and y are equinumerous. End. ☐

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Proposition 1.5. Let x, y be sets and $f : x \hookrightarrow y$ and $a \subseteq x$. Then $|f[a]| = |a|$.

Proof. $f \restriction a$ is a bijection between a and $f[a]$. $f[a]$ is a set. Hence $|a| = |f[a]|$. ☐

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Proposition 1.6. Let κ be a cardinal and $x \subseteq \kappa$. Then $|x| \leq \kappa$.

Proof. Assume $|x| > \kappa$. Then $\kappa \subseteq |x|$. Take a bijection f between $|x|$ and x. Then $f \restriction \kappa$ is an injective map from κ to x. id_x is an injective map from x to κ. Hence x and κ are equinumerous (by theorem 13.5). Indeed x is a set. Thus $|x| = \kappa$. Contradiction. ☐

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Proposition 1.7. Let x, y be sets. Then there exists an injective map from x to y iff $|x| \leq |y|$.

Proof. Case there exists an injective map from x to y. Consider an injective map f from x to y. Take a bijection g from $|x|$ to x and a bijection h from y to $|y|$. Then g is an injective map from $|x|$ to x and h is an injective map from y to $|y|$. [prover vampire] Hence $h \circ f$ is an injective map from x to $|y|$. Thus $(h \circ f) \circ g$ is an injective map from $|x|$ to $|y|$. [prover eprover] Therefore $|x| = ||x|| = |((h \circ f) \circ g)[|x|]|$. We have $((h \circ f) \circ g)[|x|] \subseteq |y|$. Hence $|x| \leq |y|$. End.

Case $|x| \leq |y|$. Take a bijection g from x to $|x|$ and a bijection h from $|y|$ to y. We have $|x| \subseteq |y|$. Hence g is an injective map from x to $|y|$. Take $f = h \circ g$. [prover vampire] Then f is an injective map from x to y. Indeed h is an injective map from $|y|$ to y. End. □

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Corollary 1.8. Let x be a set and $y \subseteq x$. Then $|y| \leq |x|$.

Proof. Define $f(v) = v$ for $v \in y$. Then f is an injective map from y to x. Hence $|y| \leq |x|$. □

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Proposition 1.9. Let x, y be nonempty sets. Then there exists a surjective map from x onto y iff $|x| \geq |y|$.

Proof. Case there exists a surjective map from x onto y. Consider a surjective map f from x onto y. Define $g(v) =$ "choose $u \in x$ such that $f(u) = v$ in u" for $v \in y$. Then g is an injective map from y to x. Indeed we can show that g is injective. Let $v, v' \in y$. Assume $g(v) = g(v')$. Take $u \in x$ such that $f(u) = v$ and $g(v) = u$. Take $u' \in x$ such that $f(u') = v'$ and $g(v') = u'$. Then $v = f(u) = f(g(v)) = f(g(v')) = f(u') = v'$. End. Hence $|x| \geq |y|$. End.

Case $|x| \geq |y|$. Then we can take an injective map f from y to x. Then f^{-1} is a bijection between $\mathrm{range}(f)$ and y. Consider an element z of y. Define

$$g(u) = \begin{cases} f^{-1}(u) & : u \in \mathrm{range}(f) \\ z & : u \notin \mathrm{range}(f) \end{cases}$$

for $u \in x$. Then g is a surjective map from x onto y. Indeed we can show that every element of y is a value of g. Let $v \in y$. Then $f(v) \in \mathrm{range}(f)$. Hence

$g(f(v)) = f^{-1}(f(v)) = v$. End. End. □

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Proposition 1.10. Let x, y be sets and $f : x \to y$ and $a \subseteq x$. Then $|f[a]| \leq |a|$.

Proof. Case a is empty. Obvious.

Case a is nonempty. $f \upharpoonright a$ is a surjective map from a onto $f[a]$ and $f[a]$ is nonempty. Hence $|f[a]| \leq |a|$ (by proposition 1.9). Indeed a and $f[a]$ are sets. End. □

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Proposition 1.11. Let x, y be nonempty sets. $|x| < |y|$ iff there exists an injective map from x to y and there exists no surjective map from x onto y.

Proof. There exists an injective map from x to y and there exists no surjective map from x onto y iff $|x| \leq |y|$ and $|x| \not\geq |y|$ (by proposition 1.7, proposition 1.9). $|x| \leq |y|$ and $|x| \not\geq |y|$ iff $|x| \leq |y|$ and $|x| \neq |y|$. $|x| \leq |y|$ and $|x| \neq |y|$ iff $|x| < |y|$. □

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Proposition 1.12. Let x, y be sets and $f : x \to y$ and $b \subseteq \mathrm{range}(f)$. Then $|f^*(b)| \geq |b|$.

Proof. Case b is empty. Obvious.

Case b is nonempty. $f \upharpoonright f^*(b)$ is a surjective map from $f^*(b)$ onto b. Hence $|f^*(b)| \geq |b|$ (by proposition 1.9). Indeed b and $f^*(b)$ are nonempty sets. End. □

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Proposition 1.13. Let x, y be sets and $f : x \hookrightarrow y$ and $b \subseteq \mathrm{range}(f)$. Then $|f^*(b)| = |b|$.

Proof. $f \upharpoonright f^*(b)$ is a bijection between $f^*(b)$ and b. Indeed $b = f[f^*(b)] = (f \upharpoonright f^*(b))[f^*(b)] = \mathrm{range}(f \upharpoonright f^*(b))$. Hence $|f^*(b)| = |b|$. □

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Proposition 1.14. Let x, y be sets such that $|y| < |x|$. Then $x \setminus y$ is nonempty.

Proof. Assume the contrary. Then $x \subseteq y$. Hence $|x| \leq |y|$. Contradiction. □

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Theorem 1.15 (Cantor). Let x be a set. Then

$$|x| < |\mathcal{P}(x)|.$$

Proof. Let us show that there exists no surjective map from x onto $\mathcal{P}(x)$. Assume the contrary. Take a surjective map f from x onto $\mathcal{P}(x)$. Define $C = \{u \in x \mid u \notin f(u)\}$. Then $C \in \mathcal{P}(x)$. Hence we can take a $u \in x$ such that $f(u) = C$. Then $u \in C$ iff $u \in f(u)$ iff $u \notin C$. Contradiction. End.

Thus $|x| \ngeq |\mathcal{P}(x)|$. Therefore $|x| < |\mathcal{P}(x)|$. □

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Theorem 1.16. For every ordinal α there exists a cardinal greater than α.

Proof. Let α be an ordinal. Take $\kappa = |\mathcal{P}(\alpha)|$. Then $\kappa > |\alpha|$.

Let us show that $\kappa > \alpha$. Assume the contrary. Then $|\mathcal{P}(\alpha)| = \kappa \leq \alpha$. Hence $\kappa = |\mathcal{P}(\alpha)| = ||\mathcal{P}(\alpha)|| \leq |\alpha|$. Contradiction. End. □