Chapter 1

Finite ordinals

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Definition 1.1.

$$\omega = \left\{ n \in \mathbf{Ord} \;\middle|\; \begin{array}{l} n \in X \text{ for every } X \subseteq \mathbf{Ord} \text{ such that } 0 \in X \text{ and for all} \\ x \in X \text{ we have } \mathrm{succ}(x) \in X \end{array} \right\}.$$

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Proposition 1.2. $0 \in \omega$.

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Proposition 1.3. Let $n \in \omega$. Then $\mathrm{succ}(n) \in \omega$.

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Proposition 1.4. Let $\Phi \subseteq \omega$. Assume that $0 \in \Phi$ and for every $x \in \Phi$ we have $\mathrm{succ}(x) \in \Phi$. Then $\Phi = \omega$.

Proof. Suppose $\Phi \neq \omega$. Consider an element n of ω that is not contained in Φ. Take $\Phi' = \Phi \setminus \{n\}$.

(1) $0 \in \Phi'$. Indeed $0 \in \Phi$ and $0 \neq n$.

(2) For each $x \in \Phi'$ we have $\mathrm{succ}(x) \in \Phi'$.
Proof. Let $x \in \Phi'$. Then $\mathrm{succ}(x) \in \Phi$.

Let us show that $\mathrm{succ}(x) \neq n$. Assume $\mathrm{succ}(x) = n$. Then $x \notin \Phi$. Indeed $n \notin \Phi$ and if $x \in \Phi$ then $n = \mathrm{succ}(x) \in \Phi$. Contradiction. End.

Thus $\mathrm{succ}(x) \in \Phi'$. Qed.

Therefore every element of ω lies in Φ'. Indeed $\Phi' \subseteq \mathbf{Ord}$. Consequently $n \in \Phi'$. Contradiction. □

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Corollary 1.5. ω is a set.

Proof. Define $f(n) = \mathrm{succ}(n)$ for $n \in \omega$. Take a subset X of ω that is inductive regarding 0 and f. Indeed f is a map from ω to ω. Then we have $0 \in X$ and for each $n \in X$ we have $\mathrm{succ}(n) \in X$. Thus $X = \omega$. Therefore ω is a set. □

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Proposition 1.6. Let $n \in \omega$. Then $n = 0$ or $n = \mathrm{succ}(m)$ for some $m \in \omega$.

Proof. Assume the contrary. Consider a $k \in \omega$ such that neither $k = 0$ nor $k = \mathrm{succ}(m)$ for some $m \in \omega$. Take a class ω' such that $\omega' = \omega \setminus \{k\}$. Then ω' is a set.

(1) $0 \in \omega'$. Indeed $k \neq 0$.

(2) For all $m \in \omega'$ we have $\mathrm{succ}(m) \in \omega'$.
Proof. Let $m \in \omega'$. Then $\mathrm{succ}(m) \neq k$. Hence $\mathrm{succ}(m) \in \omega'$. Qed.

Thus every element of ω is contained in ω'. Therefore $k \in \omega'$. Contradiction. □

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Proposition 1.7. Every element of ω is an ordinal.

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Proposition 1.8. ω is a limit ordinal.

Proof. ω is transitive.
Proof. Define $\Phi = \{n \in \omega \mid \text{for all } m \in n \text{ we have } m \in \omega\}$.

(1) $0 \in \Phi$.

(2) For all $n \in \Phi$ we have $\mathrm{succ}(n) \in \Phi$.
Proof. Let $n \in \Phi$. Then every element of n is contained in ω. Hence every element of $\mathrm{succ}(n)$ is contained in ω. Thus $\mathrm{succ}(n) \in \Phi$. Qed.

Therefore $\omega \subseteq \Phi$. Consequently ω is transitive. Qed.

Every element of ω is an ordinal. Hence every element of ω is transitive. Thus ω is an ordinal.

ω is a limit ordinal.
Proof. Assume the contrary. We have $\omega \neq 0$. Hence ω is a successor ordinal. Take an ordinal α such that $\mathrm{succ}(\alpha) = \omega$. Then $\alpha \in \omega$. Thus $\omega = \mathrm{succ}(\alpha) \in \omega$. Contradiction. Qed. □

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Proposition 1.9. Let λ be a limit ordinal. Then

$$\omega \leq \lambda.$$

Proof. Assume the contrary. Then $\lambda < \omega$. Consequently $\lambda \in \omega$. Hence $\lambda = 0$ or $\lambda = \mathrm{succ}(n)$ for some $n \in \omega$. Thus λ is not a limit ordinal. Contradiction. □

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Definition 1.10. $1 = \mathrm{succ}(0)$.

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Definition 1.11. $2 = \mathrm{succ}(1)$.

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Proposition 1.12. $1 = \{0\}$.

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Proposition 1.13. $2 = \{0, 1\}$.