Chapter 1

Transitive classes

File: `set-theory/sections/01_transitive-classes.ftl.tex`

[readtex `foundations/sections/10_sets.ftl.tex`]

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Definition 1.1. Let A be a class. A is transitive iff every element of A is a subset of A.

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Proposition 1.2. Let X be a system of sets. Then X is transitive iff for every $x \in X$ and every $y \in x$ we have $y \in X$.

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Definition 1.3. A system of transitive sets is a system of sets X such that every member of X is a transitive set.

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Proposition 1.4. Every transitive class is a system of sets.

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Proposition 1.5. Let X be a system of sets. Then X is transitive iff $\bigcup X \subseteq X$.

Proof. Case X is transitive. Let $x \in \bigcup X$. Take a member y of X such that $x \in y$. Then y is a subset of X. Hence x is an element of X. End.

Case $\bigcup X \subseteq X$. Let $x \in X$.

Let us show that $x \subseteq X$. Let $y \in x$. Then $y \in \bigcup X$. Hence $y \in X$. End. End. □

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Proposition 1.6. Let A be a transitive class. Then $\bigcup A$ is transitive.

Proof. Let $x \in \bigcup A$.

Let us show that $x \subseteq \bigcup A$. Let $y \in x$. Take a member z of A such that $x \in z$. Then $z \subseteq A$. Hence $x \in A$. Thus y is an element of some member of A. Therefore $y \in \bigcup A$. End. □

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Proposition 1.7. Let X be a system of transitive sets. Then $\bigcup X$ is transitive.

Proof. Let $x \in \bigcup X$ and $y \in x$. Take $z \in X$ such that $x \in z$. Then z is transitive. Hence $x \subseteq z$. Thus $y \in z$. Therefore $y \in \bigcup X$. □

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Proposition 1.8. Let X be a system of transitive sets. Then $X \cup \bigcup X$ is transitive.

Proof. Let $x \in X \cup \bigcup X$.

Let us show that $x \subseteq X \cup \bigcup X$. Let $u \in x$. We have $x \in X$ or $x \in \bigcup X$. If $x \in X$then $u \in \bigcup X$. If $x \in \bigcup X$ then $u \in \bigcup X$. Indeed $\bigcup X$ is transitive. Hence $u \in \bigcup X$. Thus $u \in X \cup \bigcup X$. End. □

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Proposition 1.9. Let X be a system of sets. Then X is transitive iff $X \subseteq \mathcal{P}(X)$.

Proof. Case X is transitive. Let $x \in X$. Then $x \subseteq X$. Hence $x \in \mathcal{P}(X)$. End.

Case $X \subseteq \mathcal{P}(X)$. Let $x \in X$. Then $x \in \mathcal{P}(X)$. Hence $x \subseteq X$. End. □

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Proposition 1.10. Let A be a transitive class. Then $\mathcal{P}(A)$ is transitive.

Proof. Let $x \in \mathcal{P}(A)$. Then $x \subseteq A$.

Let us show that $x \subseteq \mathcal{P}(A)$. Let $y \in x$. Then $y \in A$. Hence $y \subseteq A$. Thus $y \in \mathcal{P}(A)$. End. □