Square roots of primes are irrational

(Strictly) positive rational numbers

[synonym number/-s] [synonym divide/-s]

Signature 1. A positive rational number is an object.

Let q, s, r stand for positive rational numbers.

Signature 2. $r \cdot q$ is a positive rational number.

Axiom 3. $r \cdot q = q \cdot r$.

Axiom 4. $r \cdot (q \cdot s) = (r \cdot q) \cdot s$.

Definition 5. q is left cancellative iff for all r, s if $q \cdot s = q \cdot r$ then $s = r$.

Axiom 6. Every positive rational number is left cancellative.

Natural numbers

Signature 7. A natural number is a positive rational number.

Let m, n, k denote natural numbers.

Signature 8. 1 is a natural number.

Axiom 9. $n \cdot m$ is a natural number.

Definition 10. $n \mid m$ iff there exists k such that $k \cdot n = m$.

Let n divides m stand for $n \mid m$. Let a divisor of m stand for a natural number that divides m.

Prime numbers

Definition 11. Let p be a natural number. p is prime iff $p \neq 1$ and for all m, n if $p \mid n \cdot m$ then $p \mid n$ or $p \mid m$.

Let a prime number stand for a prime natural number.

Let p denote a prime number.

Definition 12. n and m are coprime iff n and m have no common prime divisor.

Axiom 13. There exist coprime m, n such that $m \cdot q = n$.

Let q^2 stand for $q \cdot q$.

Proposition 14. $q^2 = p$ for no positive rational number q.

Proof by contradiction. Assume the contrary. Take a positive rational number q such that $p = q^2$. Take coprime m, n such that $m \cdot q = n$. Then $p \cdot m^2 = n^2$. Therefore p divides n. Take a natural number k such that $n = k \cdot p$. Then $p \cdot m^2 = p \cdot (k \cdot n)$. Therefore $m \cdot m$ is equal to $p \cdot k^2$. Hence p divides m. Contradiction. □