Newman's lemma

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[synonym element/-s] [synonym system/-s] [synonym reduct/-s] [synonym term/-s]

Signature 1. An term is an object.

Signature 2. A rewriting system is a notion.

Let $a, b, c, d, u, v, w, x, y, z$ denote terms.

Let R denote a rewriting system.

Signature 3 (Reduct). A reduct of x in R is a term.

Let $x \to_R y$ stand for y is a reduct of x in R.

Signature 4. $x \to_R^+ y$ is a relation.

Axiom 5. $x \to_R^+ y$ iff $x \to_R y$ or there exists a term z such that $x \to_R z \to_R^+ y$.

Axiom 6. If $x \to_R^+ y \to_R^+ z$ then $x \to_R^+ z$.

Definition 7. $x \to_R^* y$ iff $x = y$ or $x \to_R^+ y$.

Lemma 8. If $x \to_R^* y \to_R^* z$ then $x \to_R^* z$.

Definition 9. R is confluent iff for all a, b, c such that $a \to_R^* b, c$ there exists d such that $b, c \to_R^* d$.

Definition 10. R is locally confluent iff for all a, b, c such that $a \to_R b, c$ there exists d such that $b, c \to_R^* d$.

Definition 11 (Terminating). R is terminating iff for all a, b such that $a \to_R^+ b$ we have $b \prec a$.

Definition 12. A normal form of x in R is a term y such that $x \to_R^* y$ and y has no reducts in R.

Lemma 13. Let R be a terminating rewriting system. Every term x has a normal form in R.

Proof by induction. $\square$

Lemma 14 (Newman). Every locally confluent terminating rewriting system is confluent.

Proof. Let R be a rewriting system. Assume R is locally confluent and terminating.

Let us demonstrate by induction that for all a, b, c such that $a \rightarrow_R^* b, c$ there exists d such that $b, c \rightarrow_R^* d$. Let a, b, c be terms. Assume $a \rightarrow_R^+ b, c$.

Take u such that $a \rightarrow_R u \rightarrow_R^* b$. Take v such that $a \rightarrow_R v \rightarrow_R^* c$. Take w such that $u, v \rightarrow_R^* w$. Take a normal form d of w in R.

$b \rightarrow_R^* d$. Indeed take x such that $b, d \rightarrow_R^* x$. $c \rightarrow_R^* d$. Indeed take y such that $c, d \rightarrow_R^* y$. End. $\square$