A Hilbert-style calculus embedded into Naproche

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The basic rules of the calculus are given as axioms. Then further derived rules are proven as propositions.

1 The implicational propositional calculus

[synonym formula/-s/-e]

Signature 1. A formula is an object.

Let P, Q, R, S denote formulas.

Signature 2. $P \supset Q$ is a formula.

Signature 3. $\vdash P$ is a relation.

Let P is deducible stand for $\vdash P$.

Axiom 4 (Detachment). Suppose $\vdash P$ and $\vdash P \supset Q$. Then $\vdash Q$.

Axiom 5 (Implosion). $\vdash P \supset (Q \supset P)$.

Axiom 6 (Chain). $\vdash (P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))$.

Proposition 7 (Weakening). (0) Assume $\vdash Q$. Then $\vdash P \supset Q$.

Proof. (1) $\vdash Q \supset (P \supset Q)$ (by Implosion).

(2) $\vdash P \supset Q$ (by Detachment, 0, 1). □

Proposition 8 (Weakening of implication). (0) Assume $\vdash Q \supset R$. Then $\vdash (P \supset Q) \supset (P \supset R)$.

Proof. (1) $\vdash P \supset (Q \supset R)$ (by Weakening, 0).

(2) $\vdash (P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))$ (by Chain).

(3) $\vdash (P \supset Q) \supset (P \supset R)$ (by Detachment, 1, 2). □

Proposition 9 (Transitivity). (0) Suppose $\vdash P \supset Q$ and $\vdash Q \supset R$. Then $\vdash P \supset R$.

Proof. (1) $\vdash (P \supset Q) \supset (P \supset R)$ (by Weakening of implication, 0).

(2) $\vdash P \supset R$ (by Detachment, 0, 1). □

Proposition 10 (Identity). $\vdash P \supset P$.

Proof. (1) $\vdash (P \supset ((P \supset P) \supset P)) \supset ((P \supset (P \supset P)) \supset (P \supset P))$ (by Chain).

(2) $\vdash P \supset ((P \supset P) \supset P)$ (by Implosion).

(3) $\vdash (P \supset (P \supset P)) \supset (P \supset P)$ (by 1, 2, Detachment).

(4) $\vdash P \supset (P \supset P)$ (by Implosion).

(5) $\vdash P \supset P$ (by 3, 4, Detachment). □

Proposition 11. (0) Suppose $\vdash P \supset (P \supset Q)$. Then $\vdash P \supset Q$.

Proof. (1) $\vdash (P \supset (P \supset Q)) \supset ((P \supset P) \supset (P \supset Q))$ (by Chain).

(2) $\vdash (P \supset P) \supset (P \supset Q)$ (by 0, 1, Detachment).

(3) $\vdash P \supset P$ (by Identity).

(4) $\vdash P \supset Q$ (by 2, 3, Detachment). □

2 Adding negation

Signature 12 (Falsum). $\bot$ is a formula.

Let ¬P stand for $P \supset \bot$.

Axiom 13 (Double negation). $\vdash \neg\neg P \supset P$.

Proposition 14 (Explosion). $\vdash \bot \supset P$.