Regularity of successor cardinals

$\mathbb{N}$aproche formalization:

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This is a formalization of a theorem of Felix Hausdorff stating that successor cardinals are always regular. On mid-range hardware $\mathbb{N}$aproche needs approximately 1:30 Minutes to verify it, plus approximately 25 minutes to verify the library files it depends on.

1 Preliminaries

[readtex `set-theory/sections/06_cardinals.ftl.tex`]

Let X denote a set. Let κ denote a cardinal.

Axiom. $|X \times X| = |X|$.

Signature. κ^+ is a cardinal such that $\kappa < \kappa^+$ and there is no cardinal ν such that $\kappa < \nu < \kappa^+$.

Axiom. $|\alpha| \leq \kappa$ for every element α of κ^+.

Definition. The constant zero on X is the function f such that $\mathrm{dom}(f) = X$ and $f(x) = 0$ for every $x \in X$.

Let 0^X stand for the constant zero on X.

2 Cofinality and regular cardinals

Definition (Cofinality). Let Y be a subset of κ. Y is cofinal in κ iff for every element x of κ there exists an element y of Y such that $x < y$.

Let a cofinal subset of κ stand for a subset of κ that is cofinal in κ.

Definition. κ is regular iff $|x| = \kappa$ for every cofinal subset x of κ.

3 Hausdorff's theorem

The following result appears in [1, p. 443], where Hausdorff mentions that the proof is *"ganz einfach"* (*"very simple"*) and can be skipped.

Theorem (Hausdorff). κ^+ is regular.
Proof by contradiction. Assume the contrary. Take a cofinal subset x of κ^+ such that $|x| \neq \kappa^+$. Then $|x| \leq \kappa$. Take a surjective map f from κ onto x (by 6.9). Indeed x and κ are nonempty and $|\kappa| = \kappa$. Then $f(\xi) \in \kappa^+$ for all $\xi \in \kappa$. For all $z \in \kappa^+$ if z has an element then there exists a surjective map h from κ onto z. Indeed κ is nonempty.

Define

$$g(z) = \begin{cases} \text{choose } h : \kappa \twoheadrightarrow z \text{ in } h & : z \text{ has an element} \\ 0^\kappa & : z \text{ has no element} \end{cases}$$

for z in κ^+.

Let us show that for all $\xi, \zeta \in \kappa$ $g(f(\xi))$ is a map such that $\zeta \in \mathrm{dom}(g(f(\xi)))$. Let $\xi, \zeta \in \kappa$. If $f(\xi)$ has an element then $g(f(\xi))$ is a surjective map from κ onto $f(\xi)$. If $f(\xi)$ has no element then $g(f(\xi)) = 0^\kappa$. Hence $\mathrm{dom}(g(f(\xi))) = \kappa$. Therefore $\zeta \in \mathrm{dom}(g(f(\xi)))$. End.

For all objects ξ, ζ we have $\xi, \zeta \in \kappa$ iff $(\xi, \zeta) \in \kappa \times \kappa$. Define $h(\xi, \zeta) = g(f(\xi))(\zeta)$ for $(\xi, \zeta) \in \kappa \times \kappa$.

Let us show that h is surjective onto κ^+.

Every element of κ^+ is an element of $h[\kappa \times \kappa]$.
Proof. Let n be an element of κ^+. Take an element ξ of κ such that $n < f(\xi)$. Take an element ζ of κ such that $g(f(\xi))(\zeta) = n$. Indeed $g(f(\xi))$ is a surjective map from κ onto $f(\xi)$. Then $n = h(\xi, \zeta)$. End.

Every element of $h[\kappa \times \kappa]$ is an element of κ^+.
Proof. Let n be an element of $h[\kappa \times \kappa]$. We can take elements a, b of κ such that $n = h(a, b)$. Then $n = g(f(a))(b)$. $f(a)$ is an element of κ^+. Every element of $f(a)$ is an element of κ^+.

Case $f(a)$ has an element. Then $g(f(a))$ is a surjective map from κ onto $f(a)$. Hence $n \in f(a) \in \kappa^+$. Thus $n \in \kappa^+$. End.

Case $f(a)$ has no element. Then $g(f(a)) = 0^\kappa$. Hence n is the empty set. Thus $n \in \kappa^+$. End. End.

Hence $\mathrm{range}(h) = h[\kappa \times \kappa] = \kappa^+$. End.

Therefore $|\kappa^+| \leq |\kappa \times \kappa|$ (by 6.9). Indeed $\kappa \times \kappa$ and κ^+ are nonempty sets and h is a surjective map from $\kappa \times \kappa$ to κ^+. Consequently $\kappa^+ \leq \kappa$. Contradiction. □

References

[1] Felix Hausdorff (1908), *Grundzüge einer Theorie der geordneten Mengen*; Teubner, *Mathematische Annalen*, vol. 65, p. 435–505