Adrian De Lon (2019–2021), Daniel Nogues Kollert (2019)
University of Bonn
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Beginnings of Tarskian geometry in Naproche (draft)

Abstract

We present a literate formalization of the beginnings of Tarskian geometry. We follow *Metamathematische Methoden in der Geometrie* by Schwabhäuser, Szmielew, and Tarski (SST), covering most of the material up to Satz 6.7, including Gupta's result that outer Pasch follows from inner Pasch. Throughout, figures help the human reader keep up with the automated theorem prover.

1. Introduction

Tarski's axiomatization of geometry is characterized by its logical elegance, relying only on two basic relations between points and a few simple axioms. These properties make Tarskian geometry attractive for formalization: in particular for research in automated deduction, see for instance Narboux (2006), Beeson and Wos (2017). For more detailed accounts of Tarski's axioms and their history see SST, Beeson (2015), and Narboux (2006).

2. The language of Tarskian geometry

The only objects under consideration are *points*. They are subject to two primitive relations: quaternary *congruence* $(-)(-)\equiv(-)(-)$ and ternary *betweenness* $(-)(-)(-)$. Congruence (also called *equidistance*) expresses that the distance between the first two points is equal to the distance of the last two points, and betweenness expresses that the second point lies between the other two on a shared line. Informally we will also talk about segments and lines, indicating them by concatenation $(-)(-)$ of points.

2.1. *Signature*. A point is an object.

2.2. *Convention*. Let a, b, c, d, e, f denote points.

2.3. *Signature* (Congruence). $ab \equiv cd$ is a relation.

2.4. *Convention*. Let $ab \not\equiv cd$ stand for it is wrong that $ab \equiv cd$.

2.5. *Signature* (Betweenness). abc is a relation.

Points are *collinear* when they lie on a single line. We will later see that betweenness is symmetric (abc implies cba), so we only need to consider three of the six permutations of three points in the definition of collinearity.

2.6. *Definition* (Collinearity). a is collinear with b and c iff abc or bca or cab.

2.7. *Axiom* (Reflexivity of congruence). We have $ab \equiv ba$.

2.8. *Axiom* (Pseudotransitivity of congruence). If $cd \equiv ab$ and $cd \equiv ef$ then $ab \equiv ef$.

2.9. *Axiom* (Identity of congruence). If $ab \equiv cc$ then $a = b$.

Segment construction allows us to extend a segment ab by a length specified by another segment de.

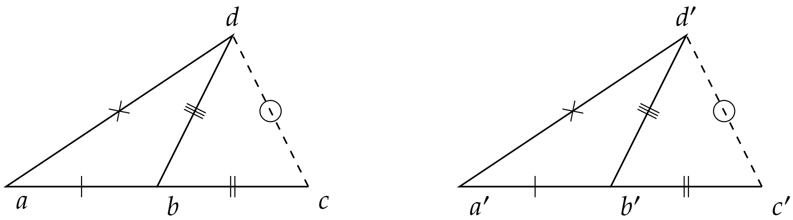
2.10. *Axiom* (Segment construction). There exists a point c such that abc and $bc \equiv de$.

We say that the points x, y, z, r, u, v, w, p are in an *outer five segment configuration* whenever OFS $\left(\begin{smallmatrix} x & y & z & r \\ u & v & w & p \end{smallmatrix}\right)$.

2.11. *Convention*. Let a', b', c', d' denote points. Let $x, y, z, u, v, w, p, q, r$ denote points.

2.12. *Definition*. OFS $\left(\begin{smallmatrix} x & y & z & r \\ u & v & w & p \end{smallmatrix}\right)$ if and only if $xyz \wedge uvw$ and we have $xy \equiv uv \wedge yz \equiv vw \wedge xr \equiv up \wedge yr \equiv vp$.

Using the concept of an outer five segment configuration, we can state the five segment axiom in a concise form.



2.13. *Axiom* (Five segment axiom). If OFS $\left(\begin{smallmatrix} a & b & c & d \\ a' & b' & c' & d' \end{smallmatrix}\right)$ and $a \neq b$ then $cd \equiv c'd'$.

2.14. *Axiom* (Identity of betweenness). If aba then $a = b$.

Tarski splits the classical axiom of Pasch into two axioms by making an inner/outer distinction, leading to logically simpler statements. We will later see that outer Pasch follows from inner Pasch, which was first demonstrated by Gupta (1965).

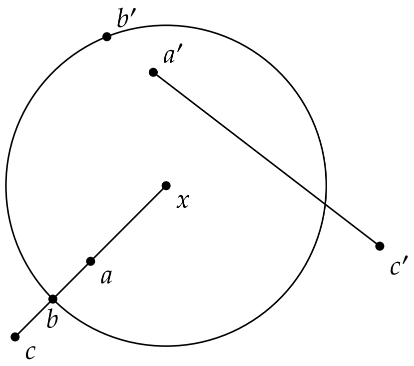
2.15. *Axiom* (Inner Pasch). If xuz and yvz then there exists a point w such that uwy and vwx.

2.16. *Axiom* (Lower dimension). There exist points α, β, γ such that α is not collinear with β and γ.

2.17. *Axiom* (Upper dimension). If $xu \equiv xv$ and $yu \equiv yv$ and $zu \equiv zv$ and $u \neq v$ then x is collinear with y and z.

2.18. *Axiom* (Euclid). Assume $x \neq r$. If xrv and yrz then there exist points s, t such that xys and xzt and svt.

Circle continuity is equivalent to the the statement *a line that has a point within a circle intersects that circle.*



2.19. *Axiom* (Circle continuity). Assume xab and xbc. Assume $xa' \equiv xa$ and $xc' \equiv xc$. Then there exists a point b' such that $xb' \equiv xb$ and $a'b'c'$.

2.20. *Lemma* (Reflexivity of congruence). For all points x, y we have $xy \equiv xy$.

2.21. *Lemma* (Symmetry of congruence). If $xy \equiv vw$ then $vw \equiv xy$.

2.22. *Lemma* (Transitivity of congruence). If $xy \equiv vw$ and $vw \equiv pq$ then $xy \equiv pq$.

2.23. *Lemma* (Congruence is independent of the order of the pairs). If $xy \equiv vw$ then $yx \equiv vw$.

2.24. *Lemma*. If $xy \equiv vw$ then $xy \equiv wv$.

2.25. *Lemma* (Zero segments are congruent). For all point x, y we have $xx \equiv yy$.

2.26. *Lemma* (Concatenation of segments). Assume xyz and rvw. Assume $xy \equiv rv$ and $yz \equiv vw$. Then $xz \equiv rw$.

Proof. We have $\mathrm{OFS}\left(\begin{smallmatrix} x & y & z & x \\ r & v & w & r \end{smallmatrix}\right)$. If $x = y$ then $r = v$. If $x \neq y$ then $xz \equiv rw$. □

2.27. *Lemma* (Uniqueness of segment construction). Assume $a \neq b$. Suppose abc and $bc \equiv de$. Suppose abc' and $bc' \equiv de$. Then $c = c'$.

Proof. We have $ac \equiv ac'$. Thus $bc \equiv bc'$. Thus $\mathrm{OFS}\left(\begin{smallmatrix} a & b & c & c \\ a & b & c & c' \end{smallmatrix}\right)$. Therefore $cc \equiv cc'$. □

2.28. *Lemma* (Right betweenness). For all points x, y we have xyy.

2.29. *Lemma* (Symmetry of betweenness). Assume xyz. Then zyx.

Left betweenness follows directly from right betweenness and symmetry of betweenness.

2.30. *Lemma* (Left betweenness). For all points x, y we have xxy.

2.31. *Lemma*. Assume xyz and yxz. Then $x = y$.

Proof. Take a point w such that ywy and xwx. Then $x = w = y$. ◻

2.32. *Lemma*. Assume xyv and yzv. Then xyz.

Proof. Take a point w such that ywy and zwx. ◻

2.33. *Lemma*. Assume xyz and yzr and $y \neq z$. Then xzr.

Proof. Take v such that xzv and $zv \equiv zr$. Then yzv and $zv \equiv zr$. Hence $v = r$. ◻

2.34. *Lemma*. Assume xyv and yzv. Then xzv.

Proof. If $y = z$ then xzv. Assume $y \neq z$. We have xyz. ◻

2.35. *Lemma*. Assume xyz and xzr. Then yzr.

2.36. *Lemma*. Assume xyz and xzr. Then xyr.

Proof. We have rzx. We have zyx. Thus ryx. Thus xyz. ◻

2.37. *Lemma*. Assume $y \neq z$. If xyz and yzr then xyr.

Existence of at least two points follows from the lower dimension axiom. All other axioms also hold in a one-point space.

2.38. *Lemma*. We have $x \neq y$ for some x, y.

2.39. *Lemma*. There exist z such that xyz and $y \neq z$.

The following follows from invoking inner Pasch twice.

2.40. *Lemma*. Assume xyz and uvz and xpu. Then there exist q such that pqz and yqv.

Proof. We have xpu and zvu. Take r such that vrx and prz. Take q such that rqz and vqy. ◻

We say that the points $a, b, c, d, a', b', c', d'$ are in an inner five segment configuration whenever IFS $\left(\begin{smallmatrix} a & b & c & d \\ a' & b' & c' & d' \end{smallmatrix}\right)$.

2.41. *Definition*. IFS $\left(\begin{smallmatrix} x & y & z & r \\ v & w & p & q \end{smallmatrix}\right)$ iff xyz and vwp and $xz \equiv vp$ and $yz \equiv wp$ and $xr \equiv vq$ and $zr \equiv pq$.

We can swap x, y with v, w.

2.42. *Lemma*. Assume IFS $\left(\begin{smallmatrix} x & y & z & r \\ v & w & p & q \end{smallmatrix}\right)$. Then $yr \equiv wq$.

Proof. Case $x \neq z$. Take points g, h such that $g \neq z$ and xzg and vph and $ph \equiv zg$. Then OFS $\begin{pmatrix} x & z & g & r \\ v & p & h & q \end{pmatrix}$. Thus $gr \equiv hq$. Thus OFS $\begin{pmatrix} g & z & y & r \\ h & p & w & q \end{pmatrix}$. Thus $yr \equiv wq$. End. □

2.43. *Lemma* (Overlapping segments). Assume xyz and rvw and $xz \equiv rw$ and $yz \equiv vw$. Then $xy \equiv rv$.

Proof. We have IFS $\left(\begin{smallmatrix} x & y & z & x \\ r & v & w & r \end{smallmatrix}\right)$. □

2.44. *Definition*. $xyz \equiv uvw$ iff $xy \equiv uv$ and $xz \equiv uw$ and $yz \equiv vw$.

2.45. *Lemma*. $xyz \equiv uvw$ iff $yxz \equiv vuw$.

2.46. *Lemma*. $xyz \equiv uvw$ iff $zyx \equiv wvu$.

2.47. *Lemma*. $xyz \equiv uvw$ iff $xzy \equiv uwv$.

If we have two congruent segments, then an inner point of one segment can be transferred congruently onto the other segment.

2.48. *Lemma*. Assume xyz and $xz \equiv rw$. Then there exists v such that rvw and $xyz \equiv rvw$.

Proof. Take u such that wru and $r \neq u$. Then take v such that urv and $rv \equiv xy$. Take a point g such that uvg and $vg \equiv yz$. Then $xz \equiv rw$. Therefore $g = w$. □

2.49. *Lemma*. Assume xyz and $xyz \equiv rvw$. Then rvw.

Proof. Take u such that ruw and $xyz \equiv ruw$. Then $ruw \equiv rvw$ and IFS $\left(\begin{smallmatrix} r & u & w & u \\ r & u & w & v \end{smallmatrix}\right)$. Then $uu \equiv uv$. Hence $u = v$. Hence rvw. □

3. Collinearity

Until now we have only used the concept of collinearity to abbreviate some axioms. We first make the straightforward observation that collinearity is invariant under permutation of the arguments.

3.1. *Lemma*. Assume that a is collinear with b and c. Then b is collinear with c and a.

3.2. *Lemma*. Assume that a is collinear with b and c. Then c is collinear with a and b.

3.3. *Lemma*. Assume that a is collinear with b and c. Then c is collinear with b and a.

3.4. *Lemma*. Assume that a is collinear with b and c. Then b is collinear with a and c.

3.5. *Lemma*. Assume that a is collinear with b and c. Then a is collinear with c and b.

Similarly, it is easy to find a common line between just two points instead of three.

3.6. *Lemma*. a is collinear with a and b for all points a, b.

3.7. *Lemma*. Assume a is collinear with b and c. Assume $ab \equiv a'b'$. Then there exists c' such that $abc \equiv a'b'c'$.

Proof. Case abc. Take c' such that $a'b'c'$ and $b'c' \equiv bc$. End. Case bac. Take c' such that $b'a'c'$ and $a'c' \equiv ac$. Then $bc \equiv b'c'$. End. Then acb. Take c' such that $a'c'b'$ and $acb \equiv a'c'b'$. □

4. Five segment configuration

4.1. *Definition*. FS $\left(\begin{smallmatrix} x & y & z & r \\ v & w & p & q \end{smallmatrix}\right)$ iff x is collinear with y and z and $xyz \equiv vwp$ and $xr \equiv vq$ and $yr \equiv wq$.

The following lemma summarizes previous statements about outer/inner five segment configurations.

4.2. *Lemma*. Assume FS $\left(\begin{smallmatrix} x & y & z & r \\ v & w & p & q \end{smallmatrix}\right)$ and $x \neq y$. Then $zr \equiv pq$.

Proof. Case xyz. We have $xyz \equiv vwp$. Thus vwp. Thus OFS $\left(\begin{smallmatrix} x & y & z & r \\ v & w & p & q \end{smallmatrix}\right)$. End. Case zxy. We have $zxy \equiv pvw$. Thus pvw. Then OFS $\left(\begin{smallmatrix} y & x & z & r \\ w & v & p & q \end{smallmatrix}\right)$. End. Then yzx. We have $yzx \equiv wpv$. Thus wpv. Then IFS $\left(\begin{smallmatrix} y & z & x & r \\ w & p & v & q \end{smallmatrix}\right)$. □

4.3. *Lemma*. Assume $x \neq y$. Assume x is collinear with y and z. Assume $xp \equiv xq$ and $yp \equiv yq$. Then $zp \equiv zq$.

Proof. We have FS $\left(\begin{smallmatrix} x & y & z & p \\ x & y & z & q \end{smallmatrix}\right)$. □

4.4. *Lemma*. Assume $a \neq b$. Assume a is collinear with b and c. Assume $ac \equiv ac'$ and $bc \equiv bc'$. Then $c' = c$.

4.5. *Lemma*. Assume xzy and $xz \equiv xp$ and $yz \equiv yp$. Then $z = p$.

Proof. Assume $x = y$. Then $x = z$ and $x = p$. Hence $z = p$. Assume $x \neq y$. □

5. Connexity of betweenness

Gupta (1965) proved that outer Pasch follows from inner Pasch. To prove Gupta's theorem, we need a few preparatory lemmas.

5.1. *Definition*. $abcd$ iff abc and abd and acd and bcd.

5.2. *Definition*. $abcde$ iff abc and abd and abe and acd and ace and ade and bcd and bce and bde and cde.

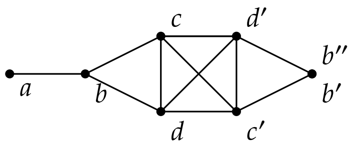
5.3. *Lemma* (Extension to quaternary betweenness). If abc and acd then $abcd$.

5.4. *Lemma* (Extension to quinary betweenness). If $abcd$ and ade then $abcde$.

5.5. *Lemma*. Assume $x \neq y$ and xyz and xyr. Then there exist points α, β such that $xr\alpha$ and $r\alpha \equiv zr$ and $xz\beta$ and $z\beta \equiv zr$.

Proof. Take point a such that xra and $ra \equiv zr$ (by segment construction). Take point b such that xzb and $zb \equiv zr$ (by segment construction). □

5.6. *Lemma.* Assume $x \neq y$ and xyz and xyr and xrp and $rp \equiv zr$ and xzq and $zq \equiv zr$. Then there exist points s, t such that zqt and rps.



5.7. *Theorem* (Outer Pasch). Assume $a \neq b$. Assume abc and abd. Then acd or adc.

Proof. Take a point c' such that adc' and $dc' \equiv cd$. Take a point d' such that acd' and $cd' \equiv cd$. Then $c = c'$ or $d = d'$.

Proof. We have $abcd'$ (by extension to quaternary betweenness). We have $abdc'$ (by extension to quaternary betweenness). Take a point b' such that $ac'b'$ and $c'b' \equiv cb$ (by segment construction). Take a point b'' such that $ad'b''$ and $d'b'' \equiv bd$ (by segment construction). Then $abcd'b''$. Then $abdc'b'$.

Thus $bc' \equiv b''c$.

Thus $bb' \equiv b''b$.

We have abb' and abb''. Thus $b'' = b'$.

We have OFS $\left(\begin{smallmatrix} b & c & d' & c' \\ b' & c' & d & c \end{smallmatrix}\right)$. Thus $c'd' \equiv cd$.

Take a point e such that cec' and ded'. Then IFS $\left(\begin{smallmatrix} d & e & d' & c \\ d & e & d' & c' \end{smallmatrix}\right)$ and IFS $\left(\begin{smallmatrix} c & e & c' & d \\ c & e & c' & d' \end{smallmatrix}\right)$. Thus $ec \equiv ec'$ and $ed \equiv ed'$.

Case $c \neq c'$. We have $c \neq d'$.

Take a point p such that $c'cp$ and $cp \equiv cd'$. Take a point r such that $d'cr$ and $cr \equiv ce$. Take a point q such that prq and $rq \equiv rp$.

Then OFS $\begin{pmatrix} d' & c & r & p \\ p & c & e & d' \end{pmatrix}$. Thus $rp \equiv ed'$. Thus $rq \equiv ed$.

Then OFS $\begin{pmatrix} d' & e & d & c \\ p & r & q & c \end{pmatrix}$. Thus $d'd \equiv pq$. Thus $cq \equiv cd$. Thus $cp \equiv cq$.

We have $rp \equiv rq$. We have $r \neq c$.

Then r is collinear with c and d'. Thus $d'p \equiv d'q$.

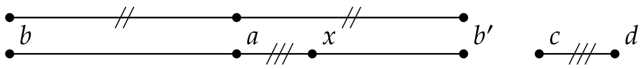
We have $c \neq d'$. Then c is collinear with d' and b. Then c is collinear with d' and b'. Thus $bp \equiv bq$ and $b'p \equiv b'q$.

Thus $b \neq b'$. Then b is collinear with c' and b'. Thus $c'p \equiv c'q$.

c' is collinear with c and p. Thus $pp \equiv pq$. Thus $p = q$. Thus $d = d'$. End. End. Therefore acd or adc. □

5.8. *Lemma.* Assume $a \neq b$. If abc and abd then bcd or bdc.

5.9. *Theorem.* If xyw and xzw then xyz or xzy.

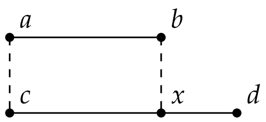


5.10. *Lemma.* Assume $a \neq b$. Then we have $ax \equiv cd$ for some point x such that abx or axb.

Proof. Take b' such that bab' and $ab' \equiv ab$. Take x such that $b'ax$ and $ax \equiv cd$. □

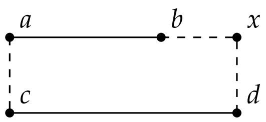
6. Comparing segments

Informally, a segment ab is smaller than a segment cd whenever we can find a subsegment cx of cd of the same length as ab.



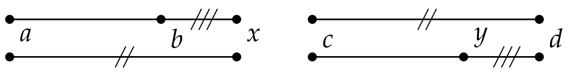
6.1. *Definition.* $ab \leq cd$ iff there exists x such that cxd and $ab \equiv cx$.

Alternatively, we can say that a segment ab is smaller than cd whenever we can extend ab to a segment ax of length cd.



6.2. *Lemma.* Assume $ab \leq cd$. Then there exists x such that abx and $ax \equiv cd$.

Proof. Take y such that cyd and $ab \equiv cy$. Take x such that abx and $bx \equiv yd$. Then $ax \equiv cd$. □



6.3. *Lemma*. Let x be a point such that abx and $ax \equiv cd$. Then $ab \leq cd$.

Proof. Take b' such that $cb'd$ and $abx \equiv cb'd$. □

6.4. *Lemma* (Transitivity of congruence and comparison). Assume $ab \equiv cd$ and $cd \leq ef$. Then $ab \leq ef$.

Proof. Take x such that exf and $cd \equiv ex$. □

6.5. *Lemma* (Transitivity of comparison and congruence). Assume $ab \leq cd$ and $cd \equiv ef$. Then $ab \leq ef$.

Proof. Take x such that abx and $ax \equiv cd$. □

6.6. *Lemma* (Reflexivity of comparison). For all points a, b we have $ab \leq ab$.

6.7. *Lemma* (Transitivity of comparison). Assume $ab \leq cd$ and $cd \leq ef$. Then $ab \leq cd$.

Proof. Take x such that abx and $ax \equiv cd$. Take y such that cdy and $cy \equiv ef$. □

6.8. *Lemma* (Antisymmetry of comparison). Assume $ab \leq cd$ and $cd \leq ab$. Then $ab \equiv cd$.

Proof. Take x such that cxd and $ab \equiv cx$. Take y such that cdy and $ab \equiv cy$. Then $cx \equiv cy$. Thus $x = d = y$. □

6.9. *Lemma* (Connexity of comparison). Let a, b, c, d be points. Then $ab \leq cd$ or $cd \leq ab$.

Proof. Case $a \neq b$. Take x such that (bax or bxa) and $bx \equiv cd$. End. □

6.10. *Lemma*. For all points a, b, c we have $aa \leq bc$.

6.11. *Lemma*. Assume abc. Then $ab \leq ac$.

6.12. *Lemma*. Assume abc. Then $bc \leq ac$.

Proof. a is a point such that cba and $ca \equiv ac$. □

6.13. *Lemma*. Assume that a is collinear with b and c. Assume $ab \leq ac$ and $bc \leq ac$. Then abc.

6.14. *Definition*. $pq < xy$ iff $pq \leq xy$ and $pq \not\equiv xy$.

6.15. *Definition*. $pq > xy$ iff $xy < pq$.

7. Rays and lines

7.1. *Definition*. a and b lie on opposite sides of u iff a, b, u are pairwise nonequal and aub.

7.2. *Definition*. a and b are equivalent with respect to u iff $a, b \neq u$ and (uab or uba).

7.3. *Convention*. Let $a \approx_u b$ stand for a and b are equivalent with respect to u.

We will see that two points are equivalent with respect to a point u iff they determine the same ray with origin u.

7.4. *Lemma*. Suppose $a, b, c \neq u$. Suppose auc. Then buc iff $a \approx_u b$.

7.5. *Lemma*. Suppose a and b are equivalent with respect to u. Then $a, b \neq u$ and there exists a point c such that $c \neq u$ and auc and buc.

7.6. *Lemma*. Suppose $a, b \neq u$. Suppose there exists a point c such that $c \neq u$ and auc and buc. Then a and b are equivalent with respect to u.

7.7. *Lemma*. a and b are equivalent with respect to u iff a is collinear with u and b and not aub.

7.8. *Lemma* (Reflexivity of relative equivalence). Suppose $a \neq u$. Then $a \approx_u a$.

7.9. *Lemma* (Symmetry of relative equivalence). If $a \approx_u b$ then $b \approx_u a$.

7.10. *Lemma* (Transitivity of relative equivalence). Assume $a \approx_u b$ and $b \approx_u c$. Then $a \approx_u c$.

Proof. Case uab. End. □

7.11. *Lemma*. Suppose $r \neq a$ and $b \neq c$. Then there exists a point x such that $x \approx_a r$ and $ax \equiv bc$.

7.12. *Lemma*. Suppose $r \neq a$ and $b \neq c$. Let x be a point such that $x \approx_a r$ and $ax \equiv bc$. Let x' be a point such that $x' \approx_a r$ and $ax' \equiv bc$. Then $x' = x$.

7.13. *Lemma*. Suppose $a \approx_u b$ and $ua \leq ub$. Then uab.

7.14. *Lemma*. Suppose $a \approx_u b$ and uab. Then $ua \leq ub$.

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