Chapter 1

Equinumerosity

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Definition 1.1. Let A, B be classes. A is equinumerous to B iff there exists a bijection between A and B.

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Proposition 1.2. Let A be a class. Then A is equinumerous to A.

Proof. id_A is a bijection between A and A. □

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Proposition 1.3. Let A, B be classes. If A and B are equinumerous then B and A are equinumerous.

Proof. Assume that A and B are equinumerous. Take a bijection f between A and B. Then f^{-1} is a bijection between B and A. Hence B and A are equinumerous. □

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Proposition 1.4. Let A, B, C be classes. If A and B are equinumerous and B and C are equinumerous then A and C are equinumerous.

Proof. Assume that A and B are equinumerous and B and C are equinumerous. Take a bijection f between A and B and a bijection g between B and C. Then $g \circ f$ is a bijection between A and C. Hence A and C are equinumerous. □

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Theorem 1.5 (Cantor-Schröder-Bernstein). Let x, y be sets. Then x and y are equinumerous iff there exists an injective map from x to y and there exists an injective map from y to x.

Proof. Case x and y are equinumerous. Take a bijection f between x and y. Then f^{-1} is a bijection between y and x. Hence f is an injective map from x to y and f^{-1} is an injective map from y to x. End.

Case there exists an injective map from x to y and there exists an injective map from y to x. Take an injective map f from x to y. Take an injective map g from y to x. We have $y \setminus f[a] \subseteq y$ for any $a \in \mathcal{P}(x)$.

(1) Define $h(a) = x \setminus g[y \setminus f[a]]$ for $a \in \mathcal{P}(x)$.

h is a map from $\mathcal{P}(x)$ to $\mathcal{P}(x)$. Indeed $h(a)$ is a subset of x for each subset a of x.

Let us show that h is subset preserving. Let u, v be subsets of x. Assume $u \subseteq v$. Then $f[u] \subseteq f[v]$. Hence $y \setminus f[v] \subseteq y \setminus f[u]$. Thus $g[y \setminus f[v]] \subseteq g[y \setminus f[u]]$. Indeed $y \setminus f[v]$ and $y \setminus f[u]$ are subsets of y. Therefore $x \setminus g[y \setminus f[u]] \subseteq x \setminus g[y \setminus f[v]]$. Consequently $h[u] \subseteq h[v]$. End.

Hence we can take a fixed point c of h (by theorem 12.4).

(2) Define $F(u) = f(u)$ for $u \in c$.

We have $c = h(c)$ iff $x \setminus c = g[y \setminus f[c]]$. g^{-1} is a bijection between $\mathrm{range}(g)$ and y. Thus $x \setminus c = g[y \setminus f[c]] \subseteq \mathrm{range}(g)$. Therefore $x \setminus c$ is a subset of $\mathrm{dom}(g^{-1})$.

(3) Define $G(u) = g^{-1}(u)$ for $u \in x \setminus c$.

F is a bijection between c and $\mathrm{range}(F)$. G is a bijection between $x \setminus c$ and $\mathrm{range}(G)$.

Define

$$H(u) = \begin{cases} F(u) & : u \in c \\ G(u) & : u \notin c \end{cases}$$

for $u \in x$.

Let us show that H is a map to y. $\mathrm{dom}(H)$ is a set and every value of H is an object.

Hence H is a map.

Let us show that every value of H lies in y. Let v be a value of H. Take $u \in x$ such that $H(u) = v$. If $u \in c$ then $v = H(u) = F(u) = f(u) \in y$. If $u \notin c$ then $v = H(u) = G(u) = g^{-1}(u) \in y$. End. End.

(4) H is surjective onto y. Indeed we can show that every element of y is a value of H. Let $v \in y$.

Case $v \in f[c]$. Take $u \in c$ such that $f(u) = v$. Then $F(u) = v$. End.

Case $v \notin f[c]$. Then $v \in y \setminus f[c]$. Hence $g(v) \in g[y \setminus f[c]]$. Thus $g(v) \in x \setminus h(c)$. We have $g(v) \in x \setminus c$. Therefore we can take $u \in x \setminus c$ such that $G(u) = v$. Then $v = H(u)$. End. End.

(5) H is injective. Indeed we can show that for all $u, v \in x$ if $u \neq v$ then $H(u) \neq H(v)$. Let $u, v \in x$. Assume $u \neq v$.

Case $u, v \in c$. Then $H(u) = F(u)$ and $H(v) = F(v)$. We have $F(u) \neq F(v)$. Hence $H(u) \neq H(v)$. End.

Case $u, v \notin c$. Then $H(u) = G(u)$ and $H(v) = G(v)$. We have $G(u) \neq G(v)$. Hence $H(u) \neq H(v)$. End.

Case $u \in c$ and $v \notin c$. Then $H(u) = F(u)$ and $H(v) = G(v)$. Hence $v \in g[y \setminus f[c]]$. We have $G(v) \in y \setminus F[c]$. Thus $G(v) \neq F(u)$. End.

Case $u \notin c$ and $v \in c$. Then $H(u) = G(u)$ and $H(v) = F(v)$. Hence $u \in g[y \setminus f[c]]$. We have $G(u) \in y \setminus f[c]$. Thus $G(u) \neq F(v)$. End. End.

Consequently H is a bijection between x and y (by 4, 5). Therefore x and y are equinumerous. End. □