Chapter 1

Fixed points

File: `foundations/sections/12_fixed-points.ftl.tex`

[readtex `foundations/sections/10_sets.ftl.tex`]

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Definition 1.1. Let f be a map. A fixed point of f is an element x of $\mathrm{dom}(f)$ such that $f(x) = x$.

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Definition 1.2. A map between systems of sets is a map from some system of sets to some system of sets.

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Definition 1.3. Let f be a map between systems of sets. f is subset preserving iff for all $x, y \in \mathrm{dom}(f)$

$$x \subseteq y \quad \text{implies} \quad f(x) \subseteq f(y).$$

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Theorem 1.4 (Knaster-Tarski). Let x be a set. Let f be a subset preserving map from $\mathcal{P}(x)$ to $\mathcal{P}(x)$. Then f has a fixed point.

Proof. (1) Define $A = \{y \mid y \subseteq x \text{ and } y \subseteq f(y)\}$. Then A is a subset of $\mathcal{P}(x)$. We have $\bigcup A \in \mathcal{P}(x)$.

Let us show that (2) $\bigcup A \subseteq f(\bigcup A)$. Let $u \in \bigcup A$. Take $y \in A$ such that $u \in y$. Then $u \in f(y)$. We have $y \subseteq \bigcup A$. Hence $f(y) \subseteq f(\bigcup A)$. Thus $f(y) \subseteq f(\bigcup A)$. Therefore $u \in f(\bigcup A)$. End.

Then $f(\bigcup A) \in A$ (by 1). Indeed $f(\bigcup A) \subseteq x$. (3) Hence $f(\bigcup A) \subseteq \bigcup A$. Indeed every element of $f(\bigcup A)$ is an element of some element of A.

Thus $f(\bigcup A) = \bigcup A$ (by 2, 3). □