Chapter 1

Binary relations

File: `foundations/sections/11_binary-relations.ftl.tex`

[readtex `foundations/sections/10_sets.ftl.tex`]

FOUNDATIONS_11_6429308924985344

Definition 1.1. A binary relation is a class R such that every element of R is a pair.

1.1 Properties of relations

Reflexivity

FOUNDATIONS_11_1126092393938944

Definition 1.2. Let R be a binary relation and A be a class. R is reflexive on A iff for all $a \in A$ we have $(a, a) \in R$.

Irreflexivity

FOUNDATIONS_11_365656446861312

Definition 1.3. Let R be a binary relation and A be a class. R is irreflexive on A iff for no $a \in A$ we have $(a, a) \in R$.

Symmetry

FOUNDATIONS_11_2056300137545728

Definition 1.4. Let R be a binary relation and A be a class. R is symmetric on A iff for all $a, b \in A$ if $(a, b) \in R$ then $(b, a) \in R$.

Antisymmetry

FOUNDATIONS_11_8301693043212288

Definition 1.5. Let R be a binary relation and A be a class. R is antisymmetric on A iff for all distinct $a, b \in A$ we have $(a, b) \notin R$ or $(b, a) \notin R$.

Asymmetry

FOUNDATIONS_11_6895428727472128

Definition 1.6. Let R be a binary relation and A be a class. R is asymmetric on A iff for all $a, b \in A$ if $(a, b) \in R$ then $(b, a) \notin R$.

Transitivity

FOUNDATIONS_11_5377309666181120

Definition 1.7. Let R be a binary relation and A be a class. R is transitive on A iff for all $a, b, c \in A$ if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

Connectedness

FOUNDATIONS_11_5902056743239680

Definition 1.8. Let R be a binary relation and A be a class. R is connected on A iff for all distinct $a, b \in A$ we have $(a, b) \in R$ or $(b, a) \in R$.

Strong connectedness

FOUNDATIONS_11_6492592562765824

Definition 1.9. Let R be a binary relation and A be a class. R is strongly connected on A iff for all $a, b \in A$ we have $(a, b) \in R$ or $(b, a) \in R$.

1.2 Order relations

Preorders.

FOUNDATIONS_11_4005024520732672

Definition 1.10. Let A be a class. A preorder on A is a binary relation that is reflexive on A and transitive on A.

Partial orders.

FOUNDATIONS_11_2162776243961856

Definition 1.11. Let A be a class. A partial order on A is a binary relation R that is reflexive on A and antisymmetric on A and transitive on A.

Let A is partially ordered by R stand for R is a partial order on A.

Strict partial orders.

FOUNDATIONS_11_4067384857985024

Definition 1.12. Let A be a class. A strict preorder on A is a binary relation that is irreflexive on A and transitive on A.

Let A is strictly preordered by R stand for R is a strict preorder on A.

FOUNDATIONS_11_5567849812721664

Proposition 1.13. Let A be a class. Any strict preorder on A is antisymmetric on A.

Let a strict partial order on A stand for a strict preorder on A. Let A is strictly partially ordered by R stand for R is a strict partial order on A.

Total orders.

FOUNDATIONS_11_5872706501214208

Definition 1.14. Let A be a class. A total order on A is a partial order on A that is connected on A.

Let A is totally ordered by R stand for R is a total order on A.

Let a linear order on A stand for a total order on A. Let A is linearly ordered by R stand for R is a linear order on A.

Strict total orders.

FOUNDATIONS_11_5840248768561152

Definition 1.15. Let A be a class. A strict total order on A is a strict partial order on A that is connected on A.

Let A is stritcly totally ordered by R stand for R is a strict total order on A.

Let a strict linear order on A stand for a strict total order on A. Let A is strictly linearly ordered by R stand for R is a strict linear order on A.

1.3 Well-founded relations

FOUNDATIONS_11_2729326472593408

Definition 1.16. Let A be a class and R be a binary relation. A least element of A regarding R is an element a of A such that there exists no $x \in A$ such that $(x, a) \in R$.

FOUNDATIONS_11_2420057567133696

Definition 1.17. Let A be a class and R be a binary relation. R is wellfounded on A iff every nonempty subclass of A has a least element regarding R.

FOUNDATIONS_11_3262141912055808

Definition 1.18. Let A be a class and R be a binary relation. R is strongly wellfounded on A iff R is wellfounded on A and for all $b \in A$ there exists a set X such that

$$X = \{a \in A \mid (a, b) \in R\}.$$

FOUNDATIONS_11_6149137814781952

Definition 1.19. Let A be a class. A wellorder on A is a strict linear order on A that is wellfounded on A.

FOUNDATIONS_11_8163723743068160

Definition 1.20. Let A be a class. A strong wellorder on A is a strict linear order on A that is strongly wellfounded on A.

1.4 Epsilon induction

FOUNDATIONS_11_4800525813940224

Definition 1.21.

$$\in = \{(a, x) \mid x \text{ is a set that contains } a\}.$$

FOUNDATIONS_11_5668859243659264

Proposition 1.22. $\in$ is strongly wellfounded on any system of sets.

Proof. Let X be a system of sets.

(1) $\in$ is wellfounded on X.
Proof. Let A be a nonempty subclass of X. Take an element x of A such that A and x are disjoint. Then x is a least element of A regarding $\in$. Indeed for any $a \in A$ if $a \in x$ then $a \in A \cap x$. Qed.

(2) For all $x \in X$ there exists a set Y such that $Y = \{y \in X \mid (y, x) \in \in\}$.
Proof. Let $x \in X$. Define $Y = \{y \in X \mid (y, x) \in \in\}$. Then $Y = \{y \in X \mid y \in x\}$. Hence Y is a subclass of x. Thus Y is a set. Qed. □

FOUNDATIONS_11_6337807438053376

Corollary 1.23. Every nonempty system of sets has a least element regarding $\in$.

FOUNDATIONS_11_2812087589928960

Proposition 1.24. Let Φ be a class. (Induction hypothesis) Assume that for all sets x if Φ contains every element of x that is a set then Φ contains x. Then Φ contains every set.

Proof. Assume the contrary. Define $M = \{x \mid x \text{ is a set such that } x \notin \Phi\}$. Then M is nonempty. Hence we can take a least element x of M regarding $\in$. Then x is a set such that every element of x that is a set is contained in Φ. Thus Φ contains x (by induction hypothesis). Contradiction. □