Chapter 1

Computation laws for images and preimages

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FOUNDATIONS_07_5919649206108160

Proposition 1.1. Let A, B be classes and $f : A \to B$ and $X \subseteq A$. Then

$$f_*(X) = \{f(x) \mid x \in X\}.$$

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Corollary 1.2. Let A, B be classes and $f : A \to B$. Then

$$f_*(A) = \mathrm{range}(f).$$

FOUNDATIONS_07_1818812171157504

Corollary 1.3. Let A, B be classes and $f : A \to B$ and $X \subseteq A$. Then

$$f_*(X) = \mathrm{range}(f \restriction X).$$

FOUNDATIONS_07_911395830890496

Proposition 1.4. Let A be a class and $X \subseteq A$. Then

$$(\mathrm{id}_A)_*(X) = X.$$

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Proposition 1.5. Let B be a class and $Y \subseteq B$. Then

$$(\mathrm{id}_B)^*(Y) = Y.$$

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Proposition 1.6. Let A, B be classes and $f : A \to B$ and $Y \subseteq B$ and $a \in A$. Then

$$f(a) \in Y \quad \text{iff} \quad a \in f^*(Y).$$

Proof. We have $f^*(Y) = \{x \in A \mid f(x) \in Y\}$. Hence $a \in f^*(Y)$ iff $a \in A$ and $f(a) \in Y$. Then we have the thesis. □

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Proposition 1.7. Let A, B be classes and $f : A \to B$. Then

$$f_*(A) \subseteq B.$$

Proof. $f_*(A) = f_*(\mathrm{dom}(f)) = \mathrm{range}(f) \subseteq B$. □

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Proposition 1.8. Let A, B be classes and $f : A \to B$. Then

$$f^*(B) = A.$$

Proof. We have $f^*(B) = \{a \in A \mid f(a) \in B\}$. $f(a) \in B$ for all $a \in A$. Hence the thesis. □

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Proposition 1.9. Let A, B be classes and $f : A \to B$. Then

$$f_*(f^*(B)) = f_*(A).$$

Proof. Let us show that $f_*(f^*(B)) \subseteq f_*(A)$. Let $b \in f_*(f^*(B))$. Take $a \in f^*(B) \cap A$ such that $b = f(a)$. Then $a \in A$. Hence $b \in f_*(A)$. End.

Let us show that $f_*(A) \subseteq f_*(f^*(B))$. Let $b \in f_*(A)$. Take $a \in A$ such that $b = f(a)$. We have $b \in B$. Hence $a \in f^*(B)$. Thus $f(a) \in f_*(f^*(B))$. Indeed $f^*(B) \subseteq A$. Therefore $b \in f_*(f^*(B))$. End. □

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Proposition 1.10. Let A, B be classes and $f : A \to B$. Then

$$f^*(f_*(A)) = A.$$

Proof. $f^*(f_*(A)) = \{a \in A \mid f(a) \in f_*(A)\}$. For all $a \in A$ we have $f(a) \in f_*(A)$. Hence every element of $f^*(f_*(A))$ is contained in A and every element of A is contained in $f^*(f_*(A))$. Thus $f^*(f_*(A)) = A$. □

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Proposition 1.11. Let A, B be classes and $f : A \to B$ and $Y \subseteq B$. Then

$$f_*(f^*(Y)) = Y \cap f_*(A).$$

Proof. Let us show that $f_*(f^*(Y)) \subseteq Y \cap f_*(A)$. Let $b \in f_*(f^*(Y))$. Take $a \in f^*(Y)$ such that $b = f(a)$. Then $f(a) \in Y \cap f_*(A)$. Hence we have $b \in Y \cap f_*(A)$. End.

Let us show that $Y \cap f_*(A) \subseteq f_*(f^*(Y))$. Let $b \in Y \cap f_*(A)$. Take $a \in A$ such that $b = f(a)$. Then $a \in f^*(Y)$. Hence $f(a) \in f_*(f^*(Y))$. End. □

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Corollary 1.12. Let A, B be classes and $f : A \to B$ and $Y \subseteq B$. Then

$$f_*(f^*(Y)) \subseteq Y.$$

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Proposition 1.13. Let A, B be classes and $f : A \to B$ and $X \subseteq A$. Then

$$f^*(f_*(X)) \supseteq X.$$

Proof. Let $a \in X$. Then $f(a) \in f_*(X)$. Hence $a \in f^*(f_*(X))$. Indeed $f_*(X) \subseteq B$. □

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Proposition 1.14. Let A, B be classes and $f : A \to B$ and $X \subseteq A$. Then

$$f_*(X) = \emptyset \quad \text{iff} \quad X = \emptyset.$$

Proof. Case $f_*(X)$ is empty. Then there is no $a \in X$ such that $f(a) \in f_*(X)$. For all $a \in X$ we have $f(a) \in f_*(X)$. Hence X is empty. End.

Case X is empty. For all $b \in f_*(X)$ we have $b = f(a)$ for some $a \in X$. There is no $a \in X$. Hence $f_*(X)$ is empty. End. □

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Proposition 1.15. Let A, B be classes and $f : A \to B$ and $Y \subseteq B$. Then

$$f^*(Y) = \emptyset \quad \text{iff} \quad Y \subseteq B \setminus f_*(A).$$

Proof. Case $f^*(Y)$ is empty. Let $b \in Y$. Then $b \in B$.

There is no $a \in A$ such that $b = f(a)$.
Proof. Assume the contrary. Take $a \in A$ such that $b = f(a)$. Then $a \in f^*(Y)$. Contradiction. Qed.

Hence $b \notin f_*(A)$. Therefore $b \in B \setminus f_*(A)$. End.

Case $Y \subseteq B \setminus f_*(A)$. Then no element of Y is an element of $f_*(A)$. Assume that $f^*(Y)$ is nonempty. Take $a \in f^*(Y)$. Then $f(a) \in Y$ and $f(a) \in f_*(A)$. Contradiction. End. □

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Proposition 1.16. Let A, B be classes and $f : A \to B$ and $X \subseteq A$ and $Y \subseteq B$. Then

$$f_*(X) \cap Y = \emptyset \quad \text{iff} \quad X \cap f^*(Y) = \emptyset.$$

Proof. Case $f_*(X) \cap Y$ is empty. Assume that $X \cap f^*(Y)$ is nonempty. Take $a \in X \cap f^*(Y)$. Then $f(a) \in f_*(X)$ and $f(a) \in Y$. Hence $f(a) \in f_*(X) \cap Y$. Contradiction. End.

Case $X \cap f^*(Y)$ is empty. Assume that $f_*(X) \cap Y$ is nonempty. Take $b \in f_*(X) \cap Y$. Consider a $a \in X$ such that $b = f(a)$. Then $a \in f^*(Y)$. Indeed $b \in Y$. Hence $a \in X \cap f^*(Y)$. Contradiction. End. □

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Proposition 1.17. Let A, B, C be classes and $f : A \to B$ and $g : B \to C$ and $X \subseteq A$. Then

$$(g \circ f)_*(X) = g_*(f_*(X)).$$

Proof. $((g \circ f)_*(X)) = \{g(f(a)) \mid a \in X\}$. $f_*(X)$ is a subclass of B. We have $g_*(f_*(X)) = \{g(b) \mid b \in f_*(X)\}$ and $f_*(X) = \{f(a) \mid a \in X\}$. Thus $g_*(f_*(X)) = \{g(f(a)) \mid a \in X\}$. Therefore $(g \circ f)_*(X) = g_*(f_*(X))$. □

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Proposition 1.18. Let A, B, C be classes and $f : A \to B$ and $g : B \to C$ and $Z \subseteq C$. Then

$$(g \circ f)^*(Z) = f^*(g^*(Z)).$$

Proof. $((g \circ f)^*(Z)) = \{a \in A \mid g(f(a)) \in Z\}$. We have $g^*(Z) = \{b \in B \mid g(b) \in Z\}$ and $f^*(g^*(Z)) = \{a \in A \mid f(a) \in g^*(Z)\}$. Hence $f^*(g^*(Z)) = \{a \in A \mid g(f(a)) \in Z\}$. Thus $(g \circ f)^*(Z) = f^*(g^*(Z))$. □

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Proposition 1.19. Let A, B be classes and $f : A \to B$ and $X, X' \subseteq A$. Then

$$X \subseteq X' \quad \text{implies} \quad f_*(X) \subseteq f_*(X').$$

Proof. Assume $X \subseteq X'$. Let $b \in f_*(X)$. Take $a \in X$ such that $f(a) = b$. Then $a \in X'$. Hence $b = f(a) \in f_*(X')$. □

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Proposition 1.20. Let A, B be classes and $f : A \to B$ and $Y, Y' \subseteq B$. Then

$$Y \subseteq Y' \quad \text{implies} \quad f^*(Y) \subseteq f^*(Y').$$

Proof. Assume $Y \subseteq Y'$. Let $a \in f^*(Y)$. Then $f(a) \in Y$. Hence $f(a) \in Y'$. Thus $a \in f^*(Y')$. □

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Proposition 1.21. Let A, B be classes and $f : A \to B$ and $X, X' \subseteq A$. Then

$$f_*(X \cup X') = f_*(X) \cup f_*(X').$$

Proof. Let us show that $f_*(X \cup X') \subseteq f_*(X) \cup f_*(X')$. Let $b \in f_*(X \cup X')$. Take $a \in X \cup X'$ such that $b = f(a)$. Then $a \in X$ or $a \in X'$. Hence $f(a) \in f_*(X)$ or $f(a) \in f_*(X')$. Thus $b = f(a) \in f_*(X) \cup f_*(X')$. End.

Let us show that $f_*(X) \cup f_*(X') \subseteq f_*(X \cup X')$. Let $b \in f_*(X) \cup f_*(X')$.

Case $b \in f_*(X)$. Take $a \in X$ such that $b = f(a)$. Then $a \in X \cup X'$. Hence $b \in f_*(X \cup X')$. End.

Case $b \in f_*(X')$. Take $a \in X'$ such that $b = f(a)$. Then $a \in X \cup X'$. Hence $b \in f_*(X \cup X')$. End. End. □

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Proposition 1.22. Let A, B be classes and $f : A \to B$ and $Y, Y' \subseteq B$. Then

$$f^*(Y \cup Y') = f^*(Y) \cup f^*(Y').$$

Proof. Let us show that $f^*(Y \cup Y') \subseteq f^*(Y) \cup f^*(Y')$. Let $a \in f^*(Y \cup Y')$. Then $f(a) \in Y \cup Y'$. Hence $f(a) \in Y$ or $f(a) \in Y'$. If $f(a) \in Y$ then $a \in f^*(Y)$. If $f(a) \in Y'$ then $a \in f^*(Y')$. Thus $a \in f^*(Y) \cup f^*(Y')$. End.

Let us show that $f^*(Y) \cup f^*(Y') \subseteq f^*(Y \cup Y')$. Let $a \in f^*(Y) \cup f^*(Y')$. Then $a \in f^*(Y)$ or $a \in f^*(Y')$. If $a \in f^*(Y)$ then $f(a) \in Y$. If $a \in f^*(Y')$ then $f(a) \in Y'$. Hence $f(a) \in Y \cup Y'$. Thus $a \in f^*(Y \cup Y')$. End. □

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Proposition 1.23. Let A, B be classes and $f : A \to B$ and $X, X' \subseteq A$. Then

$$f_*(X \cap X') \subseteq f_*(X) \cap f_*(X').$$

Proof. Let $b \in f_*(X \cap X')$. Take $a \in X \cap X'$ such that $b = f(a)$. Then $a \in X$ and $a \in X'$. Hence $f(a) \in f_*(X)$ and $f(a) \in f_*(X')$. Thus $b \in f_*(X) \cap f_*(X)$. □

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Proposition 1.24. Let A, B be classes and $f : A \to B$ and $Y, Y' \subseteq B$. Then

$$f^*(Y \cap Y') = f^*(Y) \cap f^*(Y').$$

Proof. Let us show that $f^*(Y \cap Y') \subseteq f^*(Y) \cap f^*(Y')$. Let $a \in f^*(Y \cap Y')$. Then $f(a) \in Y \cap Y'$. Hence $f(a) \in Y$ and $f(a) \in Y'$. Thus $a \in f^*(Y)$ and $a \in f^*(Y')$. Therefore $a \in f^*(Y) \cap f^*(Y')$. End.

Let us show that $f^*(Y) \cap f^*(Y') \subseteq f^*(Y \cap Y')$. Let $a \in f^*(Y) \cap f^*(Y')$. Then $a \in f^*(Y)$ and $a \in f^*(Y')$. Hence $f(a) \in Y$ and $f(a) \in Y'$. Thus $f(a) \in Y \cap Y'$. Therefore $a \in f^*(Y \cap Y')$. End. □

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Proposition 1.25. Let A, B be classes and $f : A \to B$ and $X, X' \subseteq A$. Then

$$f_*(X \setminus X') \supseteq f_*(X) \setminus f_*(X').$$

Proof. Let $b \in f_*(X) \setminus f_*(X')$. Then $b \in f_*(X)$ and $b \notin f_*(X')$. Take $a \in X$ such that $b = f(a)$. If $a \in X'$ then $b \in f_*(X')$. Hence $a \notin X'$. Thus $a \in X \setminus X'$. Therefore $b = f(a) \in f_*(X \setminus X')$. □

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Proposition 1.26. Let A, B be classes and $f : A \to B$ and $Y, Y' \subseteq B$. Then

$$f^*(Y \setminus Y') = f^*(Y) \setminus f^*(Y').$$

Proof. Let us show that $f^*(Y \setminus Y') \subseteq f^*(Y) \setminus f^*(Y')$. Let $a \in f^*(Y \setminus Y')$. Then $f(a) \in Y \setminus Y'$. Hence $f(a) \in Y$ and $f(a) \notin Y'$. Thus $a \in f^*(Y)$ and $a \notin f^*(Y')$. Therefore $a \in f^*(Y) \setminus f^*(Y')$. End.

Let us show that $f^*(Y) \setminus f^*(Y') \subseteq f^*(Y \setminus Y')$. Let $a \in f^*(Y) \setminus f^*(Y')$. Then $a \in f^*(Y)$ and $a \notin f^*(Y')$. Hence $f(a) \in Y$ and $f(a) \notin Y'$. Thus $f(a) \in Y \setminus Y'$. Therefore $a \in f^*(Y \setminus Y')$. End. $\square$