Chapter 1

Maps

File: `foundations/sections/06_maps.ftl.tex`

[readtex `foundations/sections/04_pairs-and-products.ftl.tex`]

1.1 Ranges

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Definition 1.1. Let f be a map. A value of f is an object b such that $b = f(a)$ for some $a \in \mathrm{dom}(f)$.

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Definition 1.2. Let f be a map. The range of f is

$$\{f(a) \mid a \in \mathrm{dom}(f)\}.$$

Let $\mathrm{range}(f)$ stand for the range of f.

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Proposition 1.3. Let f be a map and b be an object. b is a value of f iff $b \in \mathrm{range}(f)$.

Proof. Case b is a value of f. Take $a \in \mathrm{dom}(f)$ such that $b = f(a)$. b is an object. Hence $b \in \mathrm{range}(f)$. End.

Case $b \in \mathrm{range}(f)$. Then b is an object such that $b = f(a)$ for some $a \in \mathrm{dom}(f)$. Hence b is a value of f. End. □

1.2 The identity map

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Definition 1.4. Let A be a class. id_A is the map h such that h is defined on A and $h(a) = a$ for all $a \in A$.

Let the identity map on A stand for id_A.

1.3 Composition

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Definition 1.5. Let f, g be maps. Assume $\mathrm{range}(f) \subseteq \mathrm{dom}(g)$. $g \circ f$ is the map h such that h is defined on $\mathrm{dom}(f)$ and $h(a) = g(f(a))$ for all $a \in \mathrm{dom}(f)$.

Let the composition of g and f stand for $g \circ f$.

1.4 Restriction

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Definition 1.6. Let f be a map and $X \subseteq \mathrm{dom}(f)$. $f \upharpoonright X$ is the map h such that h is defined on X and $h(a) = f(a)$ for all $a \in X$.

Let the restriction of f to X stand for $f \upharpoonright X$.

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Proposition 1.7. Let A be a class and $X \subseteq A$. Then $\mathrm{id}_A \upharpoonright X = \mathrm{id}_X$.

1.5 Images and preimages

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Definition 1.8. Let f be a map and A be a class. The image of A under f is

$$\{f(a) \mid a \in \mathrm{dom}(f) \cap A\}.$$

Let the direct image of A under f stand for the image of A under f. Let $f_*(A)$ stand for the image of A under f.

Let $f[A]$ stand for $f_*(A)$.

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Definition 1.9. Let f be a map and B be a class. The preimage of B under f is

$$\{a \in \mathrm{dom}(f) \mid f(a) \in B\}.$$

Let the inverse image of B under f stand for the preimage of B under f. Let $f^*(B)$ stand for the preimage of B under f.

1.6 Maps between classes

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Definition 1.10. Let A be a class. A map of A is a map f such that $\mathrm{dom}(f) = A$.

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Definition 1.11. Let B be a class. A map to B is a map f such that $f(a) \in B$ for each $a \in \mathrm{dom}(f)$.

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Definition 1.12. Let A, B be classes. A map from A to B is a map f such that $\mathrm{dom}(f) = A$ and $f(a) \in B$ for each $a \in A$.

Let $f : A \to B$ stand for f is a map from A to B.

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Definition 1.13. Let A be a class. A map on A is a map from A to A.

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Proposition 1.14. Let A, B be classes and $f, g : A \to B$. Assume that $f(a) = g(a)$ for all $a \in A$. Then $f = g$.

Proposition 1.15. Let A, B be classes and f be a map of A. Assume that $f(a) \in B$ for all $a \in A$. Then f is a map from A to B iff $\mathrm{range}(f) \subseteq B$.

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Proposition 1.16. Let A be a class. Then id_A is a map on A.

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Proposition 1.17. Let A, B, C be classes and $f : A \to B$ and $g : B \to C$. Then $g \circ f : A \to C$.

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Proposition 1.18. Let A, B be classes and $f : A \to B$ and $X \subseteq A$. Then $f \upharpoonright X : X \to B$.

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Proposition 1.19. Let A, B be classes and $f : A \to B$. Then

$$f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f.$$

Proof. A is the domain of $f \circ \mathrm{id}_A$ and the domain of f and the domain of $\mathrm{id}_B \circ f$. We have $(f \circ \mathrm{id}_A)(a) = f(\mathrm{id}_A(a)) = f(a) = \mathrm{id}_B(f(a)) = (\mathrm{id}_B \circ f)(a)$ for all $a \in A$. Hence

$f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$. □

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Proposition 1.20. Let A be a class and $X \subseteq A$. Then

$$\mathrm{id}_A \restriction X = \mathrm{id}_X .$$

Proof. We have $\mathrm{dom}(\mathrm{id}_A \restriction X) = X = \mathrm{dom}(\mathrm{id}_X)$. $(\mathrm{id}_A \restriction X)(a) = \mathrm{id}_A(a) = a = \mathrm{id}_X(a)$ for all $a \in X$. Hence $\mathrm{id}_A \restriction X = \mathrm{id}_X$. □

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Proposition 1.21. Let A, B, C, D be classes and $f : A \to B$ and $g : B \to C$ and $h : C \to D$. Then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Proof. $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are maps from A to D.

Let us show that $(h \circ (g \circ f))(a) = ((h \circ g) \circ f)(a)$ for all $a \in A$. Let $a \in A$. Then $(h \circ (g \circ f))(a) = h((g \circ f)(a)) = h(g(f(a))) = (h \circ g)(f(a)) = ((h \circ g) \circ f)(a)$. End.

Hence $h \circ (g \circ f) = (h \circ g) \circ f$. □

1.7 Maps and products

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Definition 1.22. Let f be a map such that $\mathrm{dom}(f) = A \times B$ for some nonempty classes A, B. Let a be an object such that $(a, b) \in \mathrm{dom}(f)$ for some object b. $f(a, -)$ is the map such that $\mathrm{dom}(f(a, -)) = B$ and $f(a, -)(b) = f(a, b)$ for all $b \in B$ where B is the class such that $\mathrm{dom}(f) = A \times B$ for some class A.

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Definition 1.23. Let f be a map such that $\mathrm{dom}(f) = A \times B$ for some nonempty classes A, B. Let b be an object such that $(a, b) \in \mathrm{dom}(f)$ for some object a. $f(-, b)$ is the map such that $\mathrm{dom}(f(-, b)) = A$ and $f(-, b)(a) = f(a, b)$ for all $a \in A$ where A is the class such that $\mathrm{dom}(f) = A \times B$ for some class B.

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Proposition 1.24. Let A, B, C be classes such that A, B are nonempty and $a \in A$. Let f be a map from $A \times B$ to C. Then $f(a, -)$ is a map from B to C.

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Proposition 1.25. Let A, B, C be classes such that A, B are nonempty and $b \in B$. Let f be a map from $A \times B$ to C. Then $f(-, b)$ is a map from A to C.

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Proposition 1.26. Let A, B, C be classes and f be a map of $A \times B$. Assume that $f(a, b) \in C$ for all $a \in A$ and all $b \in B$. Then f is a map from $A \times B$ to C.

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Proposition 1.27. Let A, B, C be classes and f be a map from $A \times B$ to C. Let $a \in A$ and $b \in B$. Then $f(a, b) \in C$.