Chapter 1

Computation laws for Cartesian products

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[readtex `foundations/sections/02_computation-laws-for-classes.ftl.tex`]

[readtex `foundations/sections/04_pairs-and-products.ftl.tex`]

Subclass laws

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Proposition 1.1. Let A, B, C be classes. Then

$$A \subseteq B \quad \text{implies} \quad A \times C \subseteq B \times C.$$

Proof. Assume $A \subseteq B$. Let $x \in A \times C$. Take $a \in A$ and $c \in C$ such that $x = (a, c)$. Then $a \in B$. Hence $(a, c) \in B \times C$. □

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Proposition 1.2. Let A, A', B, B' be classes. Assume that A and A' are nonempty. Then

$$(A \times A') \subseteq (B \times B') \quad \text{iff} \quad (A \subseteq B \text{ and } A' \subseteq B').$$

Proof. Case $(A \times A') \subseteq (B \times B')$. Let us show that for all $a \in A$ and all $a' \in A'$ we have $a \in B$ and $a' \in B'$. Let $a \in A$ and $a' \in A'$. Then $(a, a') \in A \times A'$. Hence $(a, a') \in B \times B'$. Thus $a \in B$ and $a' \in B'$. End. End.

Case $A \subseteq B$ and $A' \subseteq B'$. Let $x \in A \times A'$. Take $a \in A$ and $a' \in A'$ such that $x = (a, a')$. Then $a \in B$ and $a' \in B'$. Hence $(a, a') \in B \times B'$. End. $\square$

Distributivity of product and union

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Proposition 1.3. Let A, B, C be classes. Then

$$(A \cup B) \times C = (A \times C) \cup (B \times C).$$

Proof. Let us show that $((A \cup B) \times C) \subseteq (A \times C) \cup (B \times C)$. Let $x \in (A \cup B) \times C$. Take $y \in A \cup B$ and $c \in C$ such that $x = (y, c)$. Then $y \in A$ or $y \in B$. If $y \in A$ then $x \in A \times C$ and if $y \in B$ then $x \in B \times C$. Hence $x \in A \times C$ or $x \in B \times C$. Thus $x \in (A \times C) \cup (B \times C)$. End.

Let us show that $((A \times C) \cup (B \times C)) \subseteq (A \cup B) \times C$. Let $x \in (A \times C) \cup (B \times C)$. Then $x \in A \times C$ or $x \in B \times C$. Take objects y, c such that $x = (y, c)$. Then ($y \in A$ or $y \in B$) and $c \in C$. Hence $y \in A \cup B$. Thus $x \in (A \cup B) \times C$. End. $\square$

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Proposition 1.4. Let A, B, C be classes. Then

$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

Proof. Let us show that $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$. Let $x \in A \times (B \cup C)$. Take $a \in A$ and $y \in B \cup C$ such that $x = (a, y)$. Then $y \in B$ or $y \in C$. Hence $x \in A \times B$ or $x \in A \times C$. Indeed if $y \in B$ then $x \in A \times B$ and if $y \in C$ then $x \in A \times C$. Thus $x \in (A \times B) \cup (A \times C)$. End.

Let us show that $((A \times B) \cup (A \times C)) \subseteq A \times (B \cup C)$. Let $x \in (A \times B) \cup (A \times C)$. Then $x \in A \times B$ or $x \in A \times C$. Take objects a, y such that $x = (a, y)$. Then $a \in A$ and ($y \in B$ or $y \in C$). Hence $x \in A \times (B \cup C)$. End. $\square$

Distributivity of product and intersection

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Proposition 1.5. Let A, B, C be classes. Then

$$(A \cap B) \times C = (A \times C) \cap (B \times C).$$

Proof. Let us show that $((A \cap B) \times C) \subseteq (A \times C) \cap (B \times C)$. Let $x \in (A \cap B) \times C$. Take $y \in A \cap B$ and $c \in C$ such that $x = (y, c)$. Then $y \in A$ and $y \in B$. Hence $x \in A \times C$ and $x \in B \times C$. Thus $x \in (A \times C) \cap (B \times C)$. End.

Let us show that $((A \times C) \cap (B \times C)) \subseteq (A \cap B) \times C$. Let $x \in (A \times C) \cap (B \times C)$. Then $x \in A \times C$ and $x \in B \times C$. Take objects y, z such that $x = (y, z)$. Then ($y \in A$ and $y \in B$) and $z \in C$. Hence $y \in A \cap B$. Thus $x \in (A \cap B) \times C$. End. □

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Proposition 1.6. Let A, B, C be classes. Then

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

Proof. Let us show that $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$. Let $x \in A \times (B \cap C)$. Take $a \in A$ and $b \in B \cap C$ such that $x = (a, b)$. Then $b \in B$ and $b \in C$. Hence $x \in A \times B$ and $x \in A \times C$. Thus $x \in (A \times B) \cap (A \times C)$. End.

Let us show that $((A \times B) \cap (A \times C)) \subseteq A \times (B \cap C)$. Let $x \in (A \times B) \cap (A \times C)$. Then $x \in A \times B$ and $x \in A \times C$. Take objects y, z such that $x = (y, z)$. Then $y \in A$ and ($z \in B$ and $z \in C$). Hence $x \in A \times (B \cap C)$. End. □

Distributivity of product and complement

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Proposition 1.7. Let A, B, C be classes. Then

$$(A \setminus B) \times C = (A \times C) \setminus (B \times C).$$

Proof. Let us show that $((A \setminus B) \times C) \subseteq (A \times C) \setminus (B \times C)$. Let $x \in (A \setminus B) \times C$. Take $a \in A \setminus B$ and $c \in C$ such that $x = (a, c)$. Then $a \in A$ and $a \notin B$. Hence $x \in A \times C$ and $x \notin B \times C$. Thus $x \in (A \times C) \setminus (B \times C)$. End.

Let us show that $((A \times C) \setminus (B \times C)) \subseteq (A \setminus B) \times C$. Let $x \in (A \times C) \setminus (B \times C)$. Then $x \in A \times C$ and $x \notin B \times C$. Take $a \in A$ and $c \in C$ such that $x = (a, c)$. Then $a \notin B$.

Indeed if $a \in B$ then $x \in B \times C$. Hence $a \in A \setminus B$. Thus $x \in (A \setminus B) \times C$. End. $\square$

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Proposition 1.8. Let A, B, C be classes. Then

$$A \times (B \setminus C) = (A \times B) \setminus (A \times C).$$

Proof. Let us show that $A \times (B \setminus C) \subseteq (A \times B) \setminus (A \times C)$. Let $x \in A \times (B \setminus C)$. Take $a \in A$ and $b \in B \setminus C$ such that $x = (a, b)$. Then $b \in B$ and $b \notin C$. Hence $x \in A \times B$ and $x \notin A \times C$. Thus $x \in (A \times B) \setminus (A \times C)$. End.

Let us show that $((A \times B) \setminus (A \times C)) \subseteq A \times (B \setminus C)$. Let $x \in (A \times B) \setminus (A \times C)$. Then $x \in A \times B$ and $x \notin A \times C$. Take objects a, b such that $x = (a, b)$. Then $a \in A$ and ($b \in B$ and $b \notin C$). Hence $x \in A \times (B \setminus C)$. End. $\square$

Equality law

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Proposition 1.9. Let A, A', B, B' be classes. Assume that A and A' are nonempty or B and B' are nonempty. Then

$$(A \times A') = (B \times B') \quad \text{iff} \quad (A = B \text{ and } A' = B').$$

Proof. Case $A \times A' = B \times B'$. Then A and A' are nonempty iff B and B' are nonempty.

Let us show that for all $a \in A$ and all $a' \in A'$ we have $a \in B$ and $a' \in B'$. Let $a \in A$ and $a' \in A'$. Then $(a, a') \in A \times A'$. Hence we can take $x \in B \times B'$ such that $x = (a, a')$. Thus $a \in B$ and $a' \in B'$. End.

Therefore $A \subseteq B$ and $A' \subseteq B'$. Indeed A and A' are nonempty.

Let us show that for all $b \in B$ and all $b' \in B'$ we have $b \in A$ and $b' \in A'$. Let $b \in B$ and $b' \in B'$. Then $(b, b') \in B \times B'$. Hence we can take $x \in A \times A'$ such that $x = (b, b')$. Thus $(b, b') \in A \times A'$. End.

Therefore $B \subseteq A$ and $B' \subseteq A'$. Indeed B and B' are nonempty. End.

Case $A = B$ and $A' = B'$. Trivial. $\square$

Intersections of products

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Proposition 1.10. Let A, A', B, B' be classes. Then

$$(A \times B) \cap (A' \times B') = (A \cap A') \times (B \cap B').$$

Proof. Let us show that $((A \times B) \cap (A' \times B')) \subseteq (A \cap A') \times (B \cap B')$. Let $x \in (A \times B) \cap (A' \times B')$. Then $x \in A \times B$ and $x \in A' \times B'$. Take objects a, b such that $x = (a, b)$. Then $a \in A, A'$ and $b \in B, B'$. Hence $a \in A \cap A'$ and $b \in B \cap B'$. Thus $x \in (A \cap A') \times (B \cap B')$. End.

Let us show that $(A \cap A') \times (B \cap B') \subseteq (A \times B) \cap (A' \times B')$. Let $x \in (A \cap A') \times (B \cap B')$. Take elements a, b such that $x = (a, b)$. Then $a \in A \cap A'$ and $b \in B \cap B'$. Hence $a \in A, A'$ and $b \in B, B'$. Thus $x \in A \times B$ and $x \in A' \times B'$. Therefore $x \in (A \times B) \cap (A' \times B')$. End. $\square$

Unions of products

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Proposition 1.11. Let A, A', B, B' be classes. Then

$$(A \times B) \cup (A' \times B') \subseteq (A \cup A') \times (B \cup B').$$

Proof. Let $x \in (A \times B) \cup (A' \times B')$. Then $x \in A \times B$ or $x \in A' \times B'$. Take objects a, b such that $x = (a, b)$. Then $(a \in A$ or $a \in A')$ and $(b \in B$ or $b \in B')$. Hence $a \in A \cup A'$ and $b \in B \cup B'$. Thus $x \in (A \cup A') \times (B \cup B')$. $\square$

Complements of products

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Proposition 1.12. Let A, A', B, B' be classes. Then

$$(A \times B) \setminus (A' \times B') = (A \times (B \setminus B')) \cup ((A \setminus A') \times B).$$

Proof. Let us show that $((A \times B) \setminus (A' \times B')) \subseteq (A \times (B \setminus B')) \cup ((A \setminus A') \times B)$. Let $x \in (A \times B) \setminus (A' \times B')$. Then $x \in A \times B$ and $x \notin A' \times B'$. Take $a \in A$ and $b \in B$ such that $x = (a, b)$. Then it is wrong that $a \in A'$ and $b \in B'$. Hence $a \notin A'$ or $b \notin B'$. Thus $a \in A \setminus A'$ or $b \in B \setminus B'$. Therefore $x \in A \times (B \setminus B')$ or $x \in (A \setminus A') \times B$. Hence we have $x \in (A \times (B \setminus B')) \cup ((A \setminus A') \times B)$. End.

Let us show that $(A \times (B \setminus B')) \cup ((A \setminus A') \times B) \subseteq (A \times B) \setminus (A' \times B')$. Let $x \in (A \times (B \setminus B')) \cup ((A \setminus A') \times B)$. Then $x \in (A \times (B \setminus B'))$ or $x \in ((A \setminus A') \times B)$. Take elements a, b such that $x = (a, b)$. Then ($a \in A$ and $b \in B \setminus B'$) or ($a \in A \setminus A'$ and $b \in B$).

Case $a \in A$ and $b \in B \setminus B'$. Then $a \in A$ and $b \in B$. Hence $x \in A \times B$. We have $b \notin B'$. Thus $x \notin A' \times B'$. Therefore $x \in (A \times B) \setminus (A' \times B')$. End.

Case $a \in A \setminus A'$ and $b \in B$. Then $a \in A$ and $b \in B$. Hence $x \in A \times B$. We have $a \notin A'$. Thus $x \notin A' \times B'$. Therefore $x \in (A \times B) \setminus (A' \times B')$. End. End. $\square$