Chapter 1

Ordered pairs and Cartesian products

File: foundations/sections/04_pairs-and-products.ftl.tex

[readtex `foundations/sections/01_classes.ftl.tex`]

1.1 Pairs

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Axiom 1.1. Let a, a', b, b' be objects. Then

$$(a,b) = (a',b') \quad \text{implies} \quad (a = a' \text{ and } b = b').$$

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Definition 1.2. A pair is an object p such that $p = (a,b)$ for some objects a, b.

Let an ordered pair stand for a pair.

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Definition 1.3. Let p be a pair. $\pi_1 p$ is the object a such that $p = (a, b)$ for some object b.

Let the first entry of p stand for $\pi_1 p$. Let the first component of p stand for the first entry of p.

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Definition 1.4. Let p be a pair. $\pi_2 p$ is the object b such that $p = (a, b)$ for some object a.

Let the second entry of p stand for $\pi_2 p$. Let the second component of p stand for the second entry of p.

1.2 Cartesian products

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Definition 1.5. Let A, B be classes. The Cartesian product of A and B is

$$\{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Let the direct product of A and B stand for the Cartesian product of A and B. Let $A \times B$ stand for the Cartesian product of A and B.

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Proposition 1.6. Let A, B be classes and a, b be objects. Then

$$(a, b) \in A \times B \quad \text{iff} \quad (a \in A \text{ and } b \in B).$$

Proof. Case $(a, b) \in A \times B$. We can take $a' \in A$ and $b' \in B$ such that $(a, b) = (a', b')$. Then $a = a'$ and $b = b'$. Hence $a \in A$ and $b \in B$. End.

Case $a \in A$ and $b \in B$. a and a are objects. Hence (a, b) is an object. Therefore $(a, b) \in A \times B$. End. □

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Proposition 1.7. Let A, B be classes. Then $A \times B$ is empty iff A is empty or B is empty.

Proof. Case $A \times B$ is empty. Assume that A and B are nonempty. Then we can take an element a of A and an element b of B. Then $(a, b) \in A \times B$. Contradiction. End.

Case A is empty or B is empty. Assume that $A \times B$ is nonempty. Then we can take an element c of $A \times B$. Then $c = (a, b)$ for some $a \in A$ and some $b \in B$. Hence A and B are nonempty. Contradiction. End. □

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Proposition 1.8. Let a, b be objects. Then

$$\{a\} \times \{b\} = \{(a, b)\}.$$

Proof. Let us show that $\{a\} \times \{b\} \subseteq \{(a, b)\}$. Let $c \in \{a\} \times \{b\}$. Take $a' \in \{a\}$ and $b' \in \{b\}$ such that $c = (a', b')$. We have $a' = a$ and $b' = b$. Hence $c = (a, b)$. Thus $c \in \{(a, b)\}$. End.

Let us show that $\{(a, b)\} \subseteq \{a\} \times \{b\}$. Let $c \in \{(a, b)\}$. Then $c = (a, b)$. We have $a \in \{a\}$ and $b \in \{b\}$. Hence $c \in \{a\} \times \{b\}$. End. □

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Proposition 1.9. Let A, A', B, B' be nonempty classes. Then

$$A \times B = A' \times B' \quad \text{implies} \quad (A = A' \text{ and } B = B').$$

Proof. Assume $A \times B = A' \times B'$.

(1) $A \subseteq A'$ and $B \subseteq B'$.
Proof. Let $a \in A$ and $b \in B$. Then $(a, b) \in A \times B$. Hence $(a, b) \in A' \times B'$. Thus $a \in A'$ and $b \in B'$. Qed.

(2) $A' \subseteq A$ and $B' \subseteq B$.
Proof. Let $a \in A'$ and $b \in B'$. Then $(a, b) \in A' \times B'$. Hence $(a, b) \in A \times B$. Thus $a \in A$ and $b \in B$. Qed. □