Chapter 1

Computation laws for classes

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Commutativity of union and intersection

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Proposition 1.1. Let A, B be classes. Then

$$A \cup B = B \cup A.$$

Proof. Let us show that $A \cup B \subseteq B \cup A$. Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. Hence $x \in B$ or $x \in A$. Thus $x \in B \cup A$. End.

Let us show that $B \cup A \subseteq A \cup B$. Let $x \in B \cup A$. Then $x \in B$ or $x \in A$. Hence $x \in A$ or $x \in B$. Thus $x \in A \cup B$. End. □

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Proposition 1.2. Let A, B be classes. Then

$$A \cap B = B \cap A.$$

Proof. Let us show that $A \cap B \subseteq B \cap A$. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Hence $x \in B$ and $x \in A$. Thus $x \in B \cap A$. End.

Let us show that $B \cap A \subseteq A \cap B$. Let $x \in B \cap A$. Then $x \in B$ and $x \in A$. Hence $x \in A$ and $x \in B$. Thus $x \in A \cap B$. End. ☐

Associativity of union and intersection

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Proposition 1.3. Let A, B, C be classes. Then

$$(A \cup B) \cup C = A \cup (B \cup C).$$

Proof. Let us show that $((A \cup B) \cup C) \subseteq A \cup (B \cup C)$. Let $x \in (A \cup B) \cup C$. Then $x \in A \cup B$ or $x \in C$. Hence $x \in A$ or $x \in B$ or $x \in C$. Thus $x \in A$ or $x \in (B \cup C)$. Therefore $x \in A \cup (B \cup C)$. End.

Let us show that $A \cup (B \cup C) \subseteq (A \cup B) \cup C$. Let $x \in A \cup (B \cup C)$. Then $x \in A$ or $x \in B \cup C$. Hence $x \in A$ or $x \in B$ or $x \in C$. Thus $x \in A \cup B$ or $x \in C$. Therefore $x \in (A \cup B) \cup C$. End. ☐

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Proposition 1.4. Let A, B, C be classes. Then

$$(A \cap B) \cap C = A \cap (B \cap C).$$

Proof. Let us show that $((A \cap B) \cap C) \subseteq A \cap (B \cap C)$. Let $x \in (A \cap B) \cap C$. Then $x \in A \cap B$ and $x \in C$. Hence $x \in A$ and $x \in B$ and $x \in C$. Thus $x \in A$ and $x \in (B \cap C)$. Therefore$x \in A \cap (B \cap C)$. End.

Let us show that $A \cap (B \cap C) \subseteq (A \cap B) \cap C$. Let $x \in A \cap (B \cap C)$. Then $x \in A$ and $x \in B \cap C$. Hence $x \in A$ and $x \in B$ and $x \in C$. Thus $x \in A \cap B$ and $x \in C$. Therefore $x \in (A \cap B) \cap C$. End. ☐

Distributivity of union and intersection

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Proposition 1.5. Let A, B, C be classes. Then

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Proof. Let us show that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. Hence $x \in A$ and ($x \in B$ or $x \in C$). Thus ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$). Therefore $x \in A \cap B$ or $x \in A \cap C$. Hence $x \in (A \cap B) \cup (A \cap C)$. End.

Let us show that $((A \cap B) \cup (A \cap C)) \subseteq A \cap (B \cup C)$. Let $x \in (A \cap B) \cup (A \cap C)$. Then $x \in A \cap B$ or $x \in A \cap C$. Hence ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$). Thus $x \in A$ and ($x \in B$ or $x \in C$). Therefore $x \in A$ and $x \in B \cup C$. Hence$x \in A \cap (B \cup C)$. End. □

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Proposition 1.6. Let A, B, C be classes. Then

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Proof. Let us show that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$. Let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. Hence $x \in A$ or ($x \in B$ and $x \in C$). Thus ($x \in A$ or $x \in B$) and ($x \in$ A or $x \in C$). Therefore $x \in A \cup B$ and $x \in A \cup C$. Hence $x \in (A \cup B) \cap (A \cup C)$. End.

Let us show that $((A \cup B) \cap (A \cup C)) \subseteq A \cup (B \cap C)$. Let $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$. Hence ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$). Thus $x \in A$ or ($x \in B$ and $x \in C$). Therefore $x \in A$ or $x \in B \cap C$. Hence $x \in A \cup (B \cap C)$. End. □

Idempocy laws for union and intersection

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Proposition 1.7. Let A be a class. Then

$$A \cup A = A.$$

Proof. $A \cup A = \{x \mid x \in A \text{ or } x \in A\}$. Hence $A \cup A = \{x \mid x \in A\}$. Thus $A \cup A = A$. □

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Proposition 1.8. Let A be a class. Then

$$A \cap A = A.$$

Proof. $A \cap A = \{x \mid x \in A \text{ and } x \in A\}$. Hence $A \cap A = \{x \mid x \in A\}$. Thus $A \cap A = A$. ☐

Distributivity of complement

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Proposition 1.9. Let A, B, C be classes. Then

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C).$$

Proof. Let us show that $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$. Let $x \in A \setminus (B \cap C)$. Then $x \in A$ and $x \notin B \cap C$. Hence it is wrong that ($x \in B$ and $x \in C$). Thus $x \notin B$ or $x \notin C$. Therefore $x \in A$ and ($x \notin B$ or $x \notin C$). Then ($x \in A$ and $x \notin B$) or ($x \in A$ and $x \notin C$). Hence $x \in A \setminus B$ or $x \in A \setminus C$. Thus $x \in (A \setminus B) \cup (A \setminus C)$. End.

Let us show that $((A \setminus B) \cup (A \setminus C)) \subseteq A \setminus (B \cap C)$. Let $x \in (A \setminus B) \cup (A \setminus C)$. Then $x \in A \setminus B$ or $x \in A \setminus C$. Hence ($x \in A$ and $x \notin B$) or ($x \in A$ and $x \notin C$). Thus $x \in A$ and ($x \notin B$ or $x \notin C$). Therefore $x \in A$ and not ($x \in B$ and $x \in C$). Then $x \in A$ and not $x \in B \cap C$. Hence $x \in A \setminus (B \cap C)$. End. ☐

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Proposition 1.10. Let A, B, C be classes. Then

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C).$$

Proof. Let us show that $A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C)$. Let $x \in A \setminus (B \cup C)$. Then $x \in A$ and $x \notin B \cup C$. Hence it is wrong that ($x \in B$ or $x \in C$). Thus $x \notin B$ and $x \notin C$. Therefore $x \in A$ and ($x \notin B$ and $x \notin C$). Then ($x \in A$ and $x \notin B$) and ($x \in A$ and $x \notin C$). Hence $x \in A \setminus B$ and $x \in A \setminus C$. Thus $x \in (A \setminus B) \cap (A \setminus C)$. End.

Let us show that $((A \setminus B) \cap (A \setminus C)) \subseteq A \setminus (B \cup C)$. Let $x \in (A \setminus B) \cap (A \setminus C)$. Then $x \in A \setminus B$ and $x \in A \setminus C$. Hence ($x \in A$ and $x \notin B$) and ($x \in A$ and $x \notin C$). Thus $x \in A$ and ($x \notin B$ and $x \notin C$). Therefore $x \in A$ and not ($x \in B$ or $x \in C$). Then $x \in A$ and not $x \in B \cup C$. Hence $x \in A \setminus (B \cup C)$. End. ☐

Subclass laws

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Proposition 1.11. Let A, B be classes. Then

$$A \subseteq A \cup B.$$

Proof. Let $x \in A$. Then $x \in A$ or $x \in B$. Hence $x \in A \cup B$. □

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Proposition 1.12. Let A, B be classes. Then

$$A \cap B \subseteq A.$$

Proof. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Hence $x \in A$. □

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Proposition 1.13. Let A, B be classes. Then

$$A \subseteq B \quad \text{iff} \quad A \cup B = B.$$

Proof. Case $A \subseteq B$.

Let us show that $A \cup B \subseteq B$. Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in A$ then $x \in B$. Hence $x \in B$. End.

Let us show that $B \subseteq A \cup B$. Let $x \in B$. Then $x \in A$ or $x \in B$. Hence $x \in A \cup B$. End. End.

Case $A \cup B = B$. Let $x \in A$. Then $x \in A$ or $x \in B$. Hence $x \in A \cup B = B$. End. □

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Proposition 1.14. Let A, B be classes. Then

$$A \subseteq B \quad \text{iff} \quad A \cap B = A.$$

Proof. Case $A \subseteq B$.

Let us show that $A \cap B \subseteq A$. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Hence $x \in A$. End.

Let us show that $A \subseteq A \cap B$. Let $x \in A$. Then $x \in B$. Hence $x \in A$ and $x \in B$. Thus $x \in A \cap B$. End. End.

Case $A \cap B = A$. Let $x \in A$. Then $x \in A \cap B$. Hence $x \in A$ and $x \in B$. Thus $x \in B$. End. ☐

Complement laws

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Proposition 1.15. Let A be a class. Then

$$A \setminus A = \emptyset.$$

Proof. $A \setminus A$ has no elements. Indeed $A \setminus A = \{x \mid x \in A \text{ and } x \notin A\}$. Hence the thesis. ☐

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Proposition 1.16. Let A be a class. Then

$$A \setminus \emptyset = A.$$

Proof. $A \setminus \emptyset = \{x \mid x \in A \text{ and } x \notin \emptyset\}$. No element is an element of $\emptyset$. Hence $A \setminus \emptyset = \{x \mid x \in A\}$. Then we have the thesis. ☐

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Proposition 1.17. Let A, B be classes. Then

$$A \setminus (A \setminus B) = A \cap B.$$

Proof. Let us show that $A \setminus (A \setminus B) \subseteq A \cap B$. Let $x \in A \setminus (A \setminus B)$. Then $x \in A$ and $x \notin A \setminus B$. Hence $x \notin A$ or $x \in B$. Thus $x \in B$. Therefore $x \in A \cap B$. End.

Let us show that $A \cap B \subseteq A \setminus (A \setminus B)$. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Hence $x \notin A$ or $x \in B$. Thus $x \notin A \setminus B$. Therefore $x \in A \setminus (A \setminus B)$. End. ☐

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Proposition 1.18. Let A, B be classes. Then

$$B \subseteq A \quad \text{iff} \quad A \setminus (A \setminus B) = B.$$

Proof. Case $B \subseteq A$. Obvious.

Case $A \setminus (A \setminus B) = B$. Then every element of B is an element of $A \setminus (A \setminus B)$. Thus every element of B is an element of A. Then we have the thesis. End. $\square$

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Proposition 1.19. Let A, B, C be classes. Then

$$A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C).$$

Proof. Let us show that $A \cap (B \setminus C) \subseteq (A \cap B) \setminus (A \cap C)$. Let $x \in A \cap (B \setminus C)$. Then $x \in A$ and $x \in B \setminus C$. Hence $x \in A$ and $x \in B$. Thus $x \in A \cap B$ and $x \notin C$. Therefore $x \notin A \cap C$. Then we have $x \in (A \cap B) \setminus (A \cap C)$. End.

Let us show that $((A \cap B) \setminus (A \cap C)) \subseteq A \cap (B \setminus C)$. Let $x \in (A \cap B) \setminus (A \cap C)$. Then $x \in A$ and $x \in B$. $x \notin A \cap C$. Hence $x \notin C$. Thus $x \in B \setminus C$. Therefore $x \in A \cap (B \setminus C)$. End. $\square$