Chapter 1

Classes

File: `foundations/sections/01_classes.ftl.tex`

1.1 Preliminaries

[readtex `vocabulary.ftl.tex`]

[readtex `axioms.ftl.tex`]

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1.2 Sub- and superclasses

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Definition 1.1. Let A be a class. A subclass of A is a class B such that every element of B is an element of A.

Let $B \subseteq A$ stand for B is a subclass of A. Let $B \subset A$ stand for $B \subseteq A$.

Let a superclass of B stand for a class A such that $B \subseteq A$. Let $B \supseteq A$ stand for B is a superclass of A. Let $B \supset A$ stand for $B \subseteq A$.

Let a proper subclass of A stand for a subclass B of A such that $B \neq A$. Let $B \subsetneq A$ stand for B is a proper subclass of A.

Let a proper superclass of B stand for a superclass A of B such that $A \neq B$. Let $B \supsetneq A$ stand for B is a proper superclass of A.

Let A includes B stand for $B \subseteq A$. Let B is included in A stand for $B \subseteq A$.

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Proposition 1.2. Let A be a class. Then

$$A \subseteq A.$$

Proof. Every element of A is contained in A. Therefore $A \subseteq A$. ☐

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Proposition 1.3. Let A, B, C be classes. Then

$$(A \subseteq B \text{ and } B \subseteq C) \quad \text{implies} \quad A \subseteq C.$$

Proof. Assume $A \subseteq B$ and $B \subseteq C$. Then every element of A is contained in B and every element of B is contained in C. Hence every element of A is contained in C. Thus $A \subseteq C$. ☐

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Proposition 1.4. Let A, B be classes. Then

$$(A \subseteq B \text{ and } B \subseteq A) \quad \text{implies} \quad A = B.$$

Proof. Assume $A \subseteq B$ and $B \subseteq A$. Then every element of A is contained in B and every element of B is contained in A. Hence $A = B$. ☐

1.3 The empty class

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Definition 1.5. Let A be a class. A is empty iff A has no elements.

Let A is nonempty stand for A is not empty.

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Definition 1.6.

$$\emptyset = \{x \mid x \neq x\}.$$

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Proposition 1.7. Let A be a class. A is empty iff $A = \emptyset$.

Proof. We can show that $\emptyset$ is empty. Indeed any element x of $\emptyset$ is nonequal to x. Hence if $A = \emptyset$ then A is empty. If A is empty then A and $\emptyset$ have no elements. Hence if A is empty then $A \subseteq \emptyset$ and $\emptyset \subseteq A$. Thus if A is empty then $A = \emptyset$. □

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Corollary 1.8. $\emptyset$ is empty.

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Corollary 1.9. Let A be a class. Then

$$\emptyset \subseteq A.$$

Proof. $\emptyset$ has no elements. Hence every element of $\emptyset$ is contained in A. □

1.4 Unordered pairs

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Definition 1.10. Let a, b be objects. The unordered pair of a and b is

$$\{x \mid x = a \text{ or } x = b\}.$$

Let $\{a, b\}$ stand for the unordered pair of a and b.

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Definition 1.11. An unordered pair is a class A such that $A = \{a, b\}$ for some distinct objects a, b.

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Definition 1.12. Let a be an object. The singleton class of a is

$$\{x \mid x = a\}.$$

Let $\{a\}$ stand for the singleton class of a.

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Definition 1.13. A singleton class is a class A such that $A = \{a\}$ for some object a.

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Proposition 1.14. Let a, a', b, b' be objects. Assume $\{a, b\} = \{a', b'\}$. Then ($a = a'$ and $b = b'$) or ($a = b'$ and $b = a'$).

Proof. We have $a = a'$ or $a = b'$. If $a = a'$ then $b = b'$. If $a = b'$ then $b = a'$. Hence ($a = a'$ and $b = b'$) or ($a = b'$ and $b = a'$). $\square$

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Corollary 1.15. Let a, a' be objects. Then

$$\{a\} = \{a'\} \quad \text{implies} \quad a = a'.$$

Definition 1.16. Let A be a class. A unique element of A is an element a of A such that for each $x \in A$ we have $x = a$.

Proposition 1.17. Let A be a class. Then A has a unique element iff $A = \{a\}$ for some object a.

1.5 Unions, intersections, complements

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Definition 1.18. Let A, B be classes. The union of A and B is

$$\{x \mid x \in A \text{ or } x \in B\}.$$

Let $A \cup B$ stand for the union of A and B.

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Definition 1.19. Let A, B be classes. The intersection of A and B is

$$\{x \mid x \in A \text{ and } x \in B\}.$$

Let $A \cap B$ stand for the intersection of A and B.

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Definition 1.20. Let A, B be classes. The complement of B in A is

$$\{x \mid x \in A \text{ and } x \notin B\}.$$

Let $A \setminus B$ stand for the complement of B in A.

1.6 Disjoint classes

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Definition 1.21. Let A, B be classes. A and B are disjoint iff A and B have no common elements.

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Proposition 1.22. Let A, B be classes. Then A and B are disjoint iff $A \cap B$ is empty.