Foundations of Mathematics

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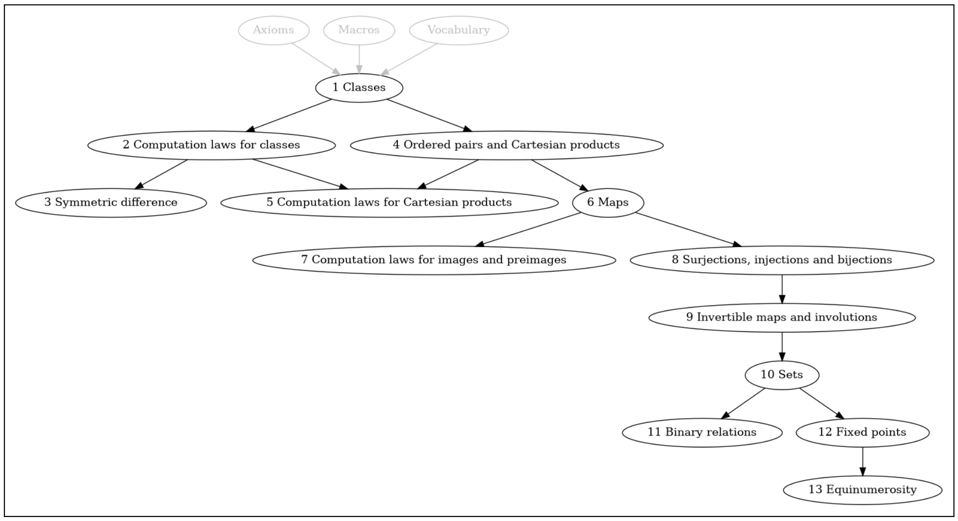
Contents



Interdependencies of the chapters

Introduction

This is a library providing a foundation of mathematics based on a Kelley-Morse like class theory with urelements. It introduces common operations on classes like unions or intersections (chapter 1) together with detailed proofs of their algebraic properties (chapter 2), the symmetric difference of two classes (chapter 3) and the notions of ordered pairs and Cartesian products (chapter 4) as well as proofs of the algebraic properties of the latter (chapter 5). Moreover, it provides common operations on maps (chapter 6), various properties of images and preimages (chapter 7) and the notions of injectivity, surjectivity, bijectivity (chapter 8) and invertibility of maps (chapter 9). The library provides an axiom system characterizing sets (chapter 10) and, furthermore, it covers the notions of binary relations (chapter 11), fixed-points of subset preserving maps (chapter 12), including and equinumerosity (chapter 13).

As two famous results it includes the Knaster-Tarski fixed point theorem (theorem 12.4) and the Cantor-Schröder-Bernstein theorem (theorem 13.5).

Usage. At the very beginning of each chapter you can find the name of its source file, e.g. `foundations/sections/01_classes.ftl.tex` for chapter 1. This filename can be used to import the chapter via ℕaproche's `readtex` instruction to another ForTheL text, e.g.:

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[readtex \path{foundations/sections/01_classes.ftl.tex}]
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Checking times. The checking times for each of the chapters may vary from computer to computer, but on mid-range hardware they are likely to be similar to those given in table below:

	Checking time	
Chapter	**without dependencies**	**with dependencies**
1	00:05 min	00:05 min
2	00:10 min	00:15 min
3	00:30 min	00:50 min
4	00:10 min	00:15 min
5	01:35 min	01:55 min
6	01:15 min	01:25 min
7	01:30 min	02:55 min
8	00:40 min	02:05 min
9	02:20 min	04:25 min
10	02:15 min	06:40 min
11	00:15 min	06:55 min
12	00:35 min	07:15 min
13	01:50 min	09:00 min

Chapter 1

Classes

File: `foundations/sections/01_classes.ftl.tex`

1.1 Preliminaries

[readtex `vocabulary.ftl.tex`]

[readtex `axioms.ftl.tex`]

[readtex `macros.ftl.tex`]

1.2 Sub- and superclasses

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Definition 1.1. Let A be a class. A subclass of A is a class B such that every element of B is an element of A.

Let $B \subseteq A$ stand for B is a subclass of A. Let $B \subset A$ stand for $B \subseteq A$.

Let a superclass of B stand for a class A such that $B \subseteq A$. Let $B \supseteq A$ stand for B is a superclass of A. Let $B \supset A$ stand for $B \subseteq A$.

Let a proper subclass of A stand for a subclass B of A such that $B \neq A$. Let $B \subsetneq A$ stand for B is a proper subclass of A.

Let a proper superclass of B stand for a superclass A of B such that $A \neq B$. Let $B \supsetneq A$ stand for B is a proper superclass of A.

Let A includes B stand for $B \subseteq A$. Let B is included in A stand for $B \subseteq A$.

FOUNDATIONS_01_5994555614691328

Proposition 1.2. Let A be a class. Then

$$A \subseteq A.$$

Proof. Every element of A is contained in A. Therefore $A \subseteq A$. □

FOUNDATIONS_01_3939677545431040

Proposition 1.3. Let A, B, C be classes. Then

$$(A \subseteq B \text{ and } B \subseteq C) \quad \text{implies} \quad A \subseteq C.$$

Proof. Assume $A \subseteq B$ and $B \subseteq C$. Then every element of A is contained in B and every element of B is contained in C. Hence every element of A is contained in C. Thus $A \subseteq C$. □

FOUNDATIONS_01_7159957847801856

Proposition 1.4. Let A, B be classes. Then

$$(A \subseteq B \text{ and } B \subseteq A) \quad \text{implies} \quad A = B.$$

Proof. Assume $A \subseteq B$ and $B \subseteq A$. Then every element of A is contained in B and every element of B is contained in A. Hence $A = B$. □

1.3 The empty class

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Definition 1.5. Let A be a class. A is empty iff A has no elements.

Let A is nonempty stand for A is not empty.

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Definition 1.6.

$$\emptyset = \{x \mid x \neq x\}.$$

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Proposition 1.7. Let A be a class. A is empty iff $A = \emptyset$.

Proof. We can show that $\emptyset$ is empty. Indeed any element x of $\emptyset$ is nonequal to x. Hence if $A = \emptyset$ then A is empty. If A is empty then A and $\emptyset$ have no elements. Hence if A is empty then $A \subseteq \emptyset$ and $\emptyset \subseteq A$. Thus if A is empty then $A = \emptyset$. □

FOUNDATIONS_01_1495141426659328

Corollary 1.8. $\emptyset$ is empty.

FOUNDATIONS_01_6931785090859008

Corollary 1.9. Let A be a class. Then

$$\emptyset \subseteq A.$$

Proof. $\emptyset$ has no elements. Hence every element of $\emptyset$ is contained in A. □

1.4 Unordered pairs

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Definition 1.10. Let a, b be objects. The unordered pair of a and b is

$$\{x \mid x = a \text{ or } x = b\}.$$

Let $\{a,\ b\}$ stand for the unordered pair of a and b.

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Definition 1.11. An unordered pair is a class A such that $A = \{a, b\}$ for some distinct objects a, b.

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Definition 1.12. Let a be an object. The singleton class of a is

$$\{x \mid x = a\}.$$

Let $\{a\}$ stand for the singleton class of a.

FOUNDATIONS_01_6786618161627136

Definition 1.13. A singleton class is a class A such that $A = \{a\}$ for some object a.

FOUNDATIONS_01_6125259604361216

Proposition 1.14. Let a, a', b, b' be objects. Assume $\{a, b\} = \{a', b'\}$. Then ($a = a'$ and $b = b'$) or ($a = b'$ and $b = a'$).

Proof. We have $a = a'$ or $a = b'$. If $a = a'$ then $b = b'$. If $a = b'$ then $b = a'$. Hence ($a = a'$ and $b = b'$) or ($a = b'$ and $b = a'$). □

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Corollary 1.15. Let a, a' be objects. Then

$$\{a\} = \{a'\} \quad \text{implies} \quad a = a'.$$

Definition 1.16. Let A be a class. A unique element of A is an element a of A such that for each $x \in A$ we have $x = a$.

Proposition 1.17. Let A be a class. Then A has a unique element iff $A = \{a\}$ for some object a.

1.5 Unions, intersections, complements

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Definition 1.18. Let A, B be classes. The union of A and B is

$$\{x \mid x \in A \text{ or } x \in B\}.$$

Let $A \cup B$ stand for the union of A and B.

FOUNDATIONS_01_5744033011859456

Definition 1.19. Let A, B be classes. The intersection of A and B is

$$\{x \mid x \in A \text{ and } x \in B\}.$$

Let $A \cap B$ stand for the intersection of A and B.

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Definition 1.20. Let A, B be classes. The complement of B in A is

$$\{x \mid x \in A \text{ and } x \notin B\}.$$

Let $A \setminus B$ stand for the complement of B in A.

1.6 Disjoint classes

FOUNDATIONS_01_4981913324355584

Definition 1.21. Let A, B be classes. A and B are disjoint iff A and B have no common elements.

FOUNDATIONS_01_1211191546347520

Proposition 1.22. Let A, B be classes. Then A and B are disjoint iff $A \cap B$ is empty.

Chapter 2

Computation laws for classes

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Commutativity of union and intersection

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Proposition 2.1. Let A, B be classes. Then

$$A \cup B = B \cup A.$$

Proof. Let us show that $A \cup B \subseteq B \cup A$. Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. Hence $x \in B$ or $x \in A$. Thus $x \in B \cup A$. End.

Let us show that $B \cup A \subseteq A \cup B$. Let $x \in B \cup A$. Then $x \in B$ or $x \in A$. Hence $x \in A$ or $x \in B$. Thus $x \in A \cup B$. End. □

FOUNDATIONS_02_7565102251245568

Proposition 2.2. Let A, B be classes. Then

$$A \cap B = B \cap A.$$

Proof. Let us show that $A \cap B \subseteq B \cap A$. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Hence $x \in B$ and $x \in A$. Thus $x \in B \cap A$. End.

Let us show that $B \cap A \subseteq A \cap B$. Let $x \in B \cap A$. Then $x \in B$ and $x \in A$. Hence $x \in A$ and $x \in B$. Thus $x \in A \cap B$. End. □

Associativity of union and intersection

FOUNDATIONS_02_3854032263184384

Proposition 2.3. Let A, B, C be classes. Then

$$(A \cup B) \cup C = A \cup (B \cup C).$$

Proof. Let us show that $((A \cup B) \cup C) \subseteq A \cup (B \cup C)$. Let $x \in (A \cup B) \cup C$. Then $x \in A \cup B$ or $x \in C$. Hence $x \in A$ or $x \in B$ or $x \in C$. Thus $x \in A$ or $x \in (B \cup C)$. Therefore $x \in A \cup (B \cup C)$. End.

Let us show that $A \cup (B \cup C) \subseteq (A \cup B) \cup C$. Let $x \in A \cup (B \cup C)$. Then $x \in A$ or $x \in B \cup C$. Hence $x \in A$ or $x \in B$ or $x \in C$. Thus $x \in A \cup B$ or $x \in C$. Therefore $x \in (A \cup B) \cup C$. End. □

FOUNDATIONS_02_906751977193472

Proposition 2.4. Let A, B, C be classes. Then

$$(A \cap B) \cap C = A \cap (B \cap C).$$

Proof. Let us show that $((A \cap B) \cap C) \subseteq A \cap (B \cap C)$. Let $x \in (A \cap B) \cap C$. Then $x \in A \cap B$ and $x \in C$. Hence $x \in A$ and $x \in B$ and $x \in C$. Thus $x \in A$ and $x \in (B \cap C)$. Therefore$x \in A \cap (B \cap C)$. End.

Let us show that $A \cap (B \cap C) \subseteq (A \cap B) \cap C$. Let $x \in A \cap (B \cap C)$. Then $x \in A$ and $x \in B \cap C$. Hence $x \in A$ and $x \in B$ and $x \in C$. Thus $x \in A \cap B$ and $x \in C$. Therefore $x \in (A \cap B) \cap C$. End. □

Distributivity of union and intersection

FOUNDATIONS_02_371139087958016

Proposition 2.5. Let A, B, C be classes. Then

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Proof. Let us show that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. Hence $x \in A$ and ($x \in B$ or $x \in C$). Thus ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$). Therefore $x \in A \cap B$ or $x \in A \cap C$. Hence $x \in (A \cap B) \cup (A \cap C)$. End.

Let us show that $((A \cap B) \cup (A \cap C)) \subseteq A \cap (B \cup C)$. Let $x \in (A \cap B) \cup (A \cap C)$. Then $x \in A \cap B$ or $x \in A \cap C$. Hence ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$). Thus $x \in A$ and ($x \in B$ or $x \in C$). Therefore $x \in A$ and $x \in B \cup C$. Hence$x \in A \cap (B \cup C)$. End. □

FOUNDATIONS_02_5937390721957888

Proposition 2.6. Let A, B, C be classes. Then

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Proof. Let us show that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$. Let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. Hence $x \in A$ or ($x \in B$ and $x \in C$). Thus ($x \in A$ or $x \in B$) and ($x \in$ A or $x \in C$). Therefore $x \in A \cup B$ and $x \in A \cup C$. Hence $x \in (A \cup B) \cap (A \cup C)$. End.

Let us show that $((A \cup B) \cap (A \cup C)) \subseteq A \cup (B \cap C)$. Let $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$. Hence ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$). Thus $x \in A$ or ($x \in B$ and $x \in C$). Therefore $x \in A$ or $x \in B \cap C$. Hence $x \in A \cup (B \cap C)$. End. □

Idempocy laws for union and intersection

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Proposition 2.7. Let A be a class. Then

$$A \cup A = A.$$

Proof. $A \cup A = \{x \mid x \in A \text{ or } x \in A\}$. Hence $A \cup A = \{x \mid x \in A\}$. Thus $A \cup A = A$. □

FOUNDATIONS_02_4053144145231872

Proposition 2.8. Let A be a class. Then

$$A \cap A = A.$$

Proof. $A \cap A = \{x \mid x \in A \text{ and } x \in A\}$. Hence $A \cap A = \{x \mid x \in A\}$. Thus $A \cap A = A$. □

Distributivity of complement

FOUNDATIONS_02_5296031436636160

Proposition 2.9. Let A, B, C be classes. Then

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C).$$

Proof. Let us show that $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$. Let $x \in A \setminus (B \cap C)$. Then $x \in A$ and $x \notin B \cap C$. Hence it is wrong that ($x \in B$ and $x \in C$). Thus $x \notin B$ or $x \notin C$. Therefore $x \in A$ and ($x \notin B$ or $x \notin C$). Then ($x \in A$ and $x \notin B$) or ($x \in A$ and $x \notin C$). Hence $x \in A \setminus B$ or $x \in A \setminus C$. Thus $x \in (A \setminus B) \cup (A \setminus C)$. End.

Let us show that $((A \setminus B) \cup (A \setminus C)) \subseteq A \setminus (B \cap C)$. Let $x \in (A \setminus B) \cup (A \setminus C)$. Then $x \in A \setminus B$ or $x \in A \setminus C$. Hence ($x \in A$ and $x \notin B$) or ($x \in A$ and $x \notin C$). Thus $x \in A$ and ($x \notin B$ or $x \notin C$). Therefore $x \in A$ and not ($x \in B$ and $x \in C$). Then $x \in A$ and not $x \in B \cap C$. Hence $x \in A \setminus (B \cap C)$. End. □

FOUNDATIONS_02_2909554153095168

Proposition 2.10. Let A, B, C be classes. Then

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C).$$

Proof. Let us show that $A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C)$. Let $x \in A \setminus (B \cup C)$. Then $x \in A$ and $x \notin B \cup C$. Hence it is wrong that ($x \in B$ or $x \in C$). Thus $x \notin B$ and $x \notin C$. Therefore $x \in A$ and ($x \notin B$ and $x \notin C$). Then ($x \in A$ and $x \notin B$) and ($x \in A$ and $x \notin C$). Hence $x \in A \setminus B$ and $x \in A \setminus C$. Thus $x \in (A \setminus B) \cap (A \setminus C)$. End.

Let us show that $((A \setminus B) \cap (A \setminus C)) \subseteq A \setminus (B \cup C)$. Let $x \in (A \setminus B) \cap (A \setminus C)$. Then $x \in A \setminus B$ and $x \in A \setminus C$. Hence ($x \in A$ and $x \notin B$) and ($x \in A$ and $x \notin C$). Thus $x \in A$ and ($x \notin B$ and $x \notin C$). Therefore $x \in A$ and not ($x \in B$ or $x \in C$). Then $x \in A$ and not $x \in B \cup C$. Hence $x \in A \setminus (B \cup C)$. End. □

Subclass laws

FOUNDATIONS_02_3793981508943872

Proposition 2.11. Let A, B be classes. Then

$$A \subseteq A \cup B.$$

Proof. Let $x \in A$. Then $x \in A$ or $x \in B$. Hence $x \in A \cup B$. □

FOUNDATIONS_02_1591517646946304

Proposition 2.12. Let A, B be classes. Then

$$A \cap B \subseteq A.$$

Proof. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Hence $x \in A$. □

FOUNDATIONS_02_6657236858306560

Proposition 2.13. Let A, B be classes. Then

$$A \subseteq B \quad \text{iff} \quad A \cup B = B.$$

Proof. Case $A \subseteq B$.

Let us show that $A \cup B \subseteq B$. Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in A$ then $x \in B$. Hence $x \in B$. End.

Let us show that $B \subseteq A \cup B$. Let $x \in B$. Then $x \in A$ or $x \in B$. Hence $x \in A \cup B$. End. End.

Case $A \cup B = B$. Let $x \in A$. Then $x \in A$ or $x \in B$. Hence $x \in A \cup B = B$. End. □

FOUNDATIONS_02_2356449346846720

Proposition 2.14. Let A, B be classes. Then

$$A \subseteq B \quad \text{iff} \quad A \cap B = A.$$

Proof. Case $A \subseteq B$.

Let us show that $A \cap B \subseteq A$. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Hence $x \in A$. End.

Let us show that $A \subseteq A \cap B$. Let $x \in A$. Then $x \in B$. Hence $x \in A$ and $x \in B$. Thus $x \in A \cap B$. End. End.

Case $A \cap B = A$. Let $x \in A$. Then $x \in A \cap B$. Hence $x \in A$ and $x \in B$. Thus $x \in B$. End. □

Complement laws

FOUNDATIONS_02_7433299337150464

Proposition 2.15. Let A be a class. Then

$$A \setminus A = \emptyset.$$

Proof. $A \setminus A$ has no elements. Indeed $A \setminus A = \{x \mid x \in A \text{ and } x \notin A\}$. Hence the thesis. □

FOUNDATIONS_02_3783696985358336

Proposition 2.16. Let A be a class. Then

$$A \setminus \emptyset = A.$$

Proof. $A \setminus \emptyset = \{x \mid x \in A \text{ and } x \notin \emptyset\}$. No element is an element of $\emptyset$. Hence $A \setminus \emptyset = \{x \mid x \in A\}$. Then we have the thesis. □

FOUNDATIONS_02_7083929257377792

Proposition 2.17. Let A, B be classes. Then

$$A \setminus (A \setminus B) = A \cap B.$$

Proof. Let us show that $A \setminus (A \setminus B) \subseteq A \cap B$. Let $x \in A \setminus (A \setminus B)$. Then $x \in A$ and $x \notin A \setminus B$. Hence $x \notin A$ or $x \in B$. Thus $x \in B$. Therefore $x \in A \cap B$. End.

Let us show that $A \cap B \subseteq A \setminus (A \setminus B)$. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Hence $x \notin A$ or $x \in B$. Thus $x \notin A \setminus B$. Therefore $x \in A \setminus (A \setminus B)$. End. □

FOUNDATIONS_02_4938646769631232

Proposition 2.18. Let A, B be classes. Then

$$B \subseteq A \quad \text{iff} \quad A \setminus (A \setminus B) = B.$$

Proof. Case $B \subseteq A$. Obvious.

Case $A \setminus (A \setminus B) = B$. Then every element of B is an element of $A \setminus (A \setminus B)$. Thus every element of B is an element of A. Then we have the thesis. End. □

FOUNDATIONS_02_5811954316738560

Proposition 2.19. Let A, B, C be classes. Then

$$A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C).$$

Proof. Let us show that $A \cap (B \setminus C) \subseteq (A \cap B) \setminus (A \cap C)$. Let $x \in A \cap (B \setminus C)$. Then $x \in A$ and $x \in B \setminus C$. Hence $x \in A$ and $x \in B$. Thus $x \in A \cap B$ and $x \notin C$. Therefore $x \notin A \cap C$. Then we have $x \in (A \cap B) \setminus (A \cap C)$. End.

Let us show that $((A \cap B) \setminus (A \cap C)) \subseteq A \cap (B \setminus C)$. Let $x \in (A \cap B) \setminus (A \cap C)$. Then $x \in A$ and $x \in B$. $x \notin A \cap C$. Hence $x \notin C$. Thus $x \in B \setminus C$. Therefore $x \in A \cap (B \setminus C)$. End. □

Chapter 3

Symmetric difference

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3.1 Definitions

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Definition 3.1. Let A, B be classes.

$$A \triangle B = (A \cup B) \setminus (A \cap B).$$

Let the symmetric difference of A and B stand for $A \triangle B$.

FOUNDATIONS_03_4886447211413504

Proposition 3.2. Let A, B be classes. Then

$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$

Proof. Let us show that $A \triangle B \subseteq (A \setminus B) \cup (B \setminus A)$. Let $u \in A \triangle B$. Then $u \in A \cup B$ and $u \notin A \cap B$. Hence ($u \in A$ or $u \in B$) and not ($u \in A$ and $u \in B$). Thus ($u \in A$ or $u \in B$) and ($u \notin A$ or $u \notin B$). Therefore if $u \in A$ then $u \notin B$. If $u \in B$ then $u \notin A$. Then we have ($u \in A$ and $u \notin B$) or ($u \in B$ and $u \notin A$). Hence $u \in A \setminus B$ or

$u \in B \setminus A$. Thus $u \in (A \setminus B) \cup (B \setminus A)$. End.

Let us show that $((A \setminus B) \cup (B \setminus A)) \subseteq A \triangle B$. Let $u \in (A \setminus B) \cup (B \setminus A)$. Then ($u \in A$ and $u \notin B$) or ($u \in B$ and $u \notin A$). If $u \in A$ and $u \notin B$ then $u \in A \cup B$ and $u \notin A \cap B$. If $u \in B$ and $u \notin A$ then $u \in A \cup B$ and $u \notin A \cap B$. Hence $u \in A \cup B$ and $u \notin A \cap B$. Thus $u \in (A \cup B) \setminus (A \cap B) = A \triangle B$. End. □

3.2 Computation laws

Commutativity

FOUNDATIONS_03_4518372049944576

Proposition 3.3. Let A, B be classes. Then

$$A \triangle B = B \triangle A.$$

Proof. $A \triangle B = (A \cup B) \setminus (A \cap B) = (B \cup A) \setminus (B \cap A) = B \triangle A$. □

Associativity

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Proposition 3.4. Let A, B, C be classes. Then

$$(A \triangle B) \triangle C = A \triangle (B \triangle C).$$

Proof. Take a class X such that $X = (((A \setminus B) \cup (B \setminus A)) \setminus C) \cup (C \setminus ((A \setminus B) \cup (B \setminus A)))$.

Take a class Y such that $Y = (A \setminus ((B \setminus C) \cup (C \setminus B))) \cup (((B \setminus C) \cup (C \setminus B)) \setminus A)$.

We have $A \triangle B = (A \setminus B) \cup (B \setminus A)$ and $B \triangle C = (B \setminus C) \cup (C \setminus B)$. Hence $(A \triangle B) \triangle C = X$ and $A \triangle (B \triangle C) = Y$.

Let us show that (I) $X \subseteq Y$. Let $x \in X$.

(I 1) Case $x \in ((A \setminus B) \cup (B \setminus A)) \setminus C$. Then $x \notin C$.

(I 1a) Case $x \in A \setminus B$. Then $x \notin B \setminus C$ and $x \notin C \setminus B$. $x \in A$. Hence $x \in A \setminus ((B \setminus C) \cup (C \setminus B))$. Thus $x \in Y$. End.

(I 1b) Case $x \in B \setminus A$. Then $x \in B \setminus C$. Hence $x \in (B \setminus C) \cup (C \setminus B)$. $x \notin A$. Thus $x \in ((B \setminus C) \cup (C \setminus B)) \setminus A$. Therefore $x \in Y$. End. End.

(I 2) Case $x \in C \setminus ((A \setminus B) \cup (B \setminus A))$. Then $x \in C$. $x \notin A \setminus B$ and $x \notin B \setminus A$. Hence not ($x \in A \setminus B$ or $x \in B \setminus A$). Thus not (($x \in A$ and $x \notin B$) or ($x \in B$ and $x \notin A$)). Therefore ($x \notin A$ or $x \in B$) and ($x \notin B$ or $x \in A$).

(I 2a) Case $x \in A$. Then $x \in B$. Hence $x \notin (B \setminus C) \cup (C \setminus B)$. Thus $x \in A \setminus ((B \setminus C) \cup (C \setminus B))$. Therefore $x \in Y$. End.

(I 2b) Case $x \notin A$. Then $x \notin B$. Hence $x \in C \setminus B$. Thus $x \in (B \setminus C) \cup (C \setminus B)$. Therefore $x \in ((B \setminus C) \cup (C \setminus B)) \setminus A$. Then we have $x \in Y$. End. End. End.

Let us show that (II) $Y \subseteq X$. Let $y \in Y$.

(II 1) Case $y \in A \setminus ((B \setminus C) \cup (C \setminus B))$. Then $y \in A$. $y \notin B \setminus C$ and $y \notin C \setminus B$. Hence not ($y \in B \setminus C$ or $y \in C \setminus B$). Thus not (($y \in B$ and $y \notin C$) or ($y \in C$ and $y \notin B$)). Therefore ($y \notin B$ or $y \in C$) and ($y \notin C$ or $y \in B$).

(II 1a) Case $y \in B$. Then $y \in C$. $y \notin A \setminus B$ and $y \notin B \setminus A$. Hence $y \notin (A \setminus B) \cup (B \setminus A)$. Thus $y \in C \setminus ((A \setminus B) \cup (B \setminus A))$. Therefore $y \in X$. End.

(II 1b) Case $y \notin B$. Then $y \notin C$. $y \in A \setminus B$. Hence $y \in (A \setminus B) \cup (B \setminus A)$. Thus $y \in ((A \setminus B) \cup (B \setminus A)) \setminus C$. Therefore $y \in X$. End. End.

(II 2) Case $y \in ((B \setminus C) \cup (C \setminus B)) \setminus A$. Then $y \notin A$.

(II 2a) Case $y \in B \setminus C$. Then $y \in B \setminus A$. Hence $y \in (A \setminus B) \cup (B \setminus A)$. Thus $y \in ((A \setminus B) \cup (B \setminus A)) \setminus C$. Therefore $y \in X$. End.

(II 2b) Case $y \in C \setminus B$. Then $y \in C$. $y \notin A \setminus B$ and $y \notin B \setminus A$. Hence $y \notin (A \setminus B) \cup (B \setminus A)$. Thus $y \in C \setminus ((A \setminus B) \cup (B \setminus A))$. Therefore $y \in X$. End. End. End. □

Distributivity

FOUNDATIONS_03_4119141910839296

Proposition 3.5. Let A, B, C be classes. Then

$$A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C).$$

Proof. $A \cap (B \triangle C) = A \cap ((B \setminus C) \cup (C \setminus B)) = (A \cap (B \setminus C)) \cup (A \cap (C \setminus B))$.

$A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$. $A \cap (C \setminus B) = (A \cap C) \setminus (A \cap B)$.

Hence $A \cap (B \triangle C) = ((A \cap B) \setminus (A \cap C)) \cup ((A \cap C) \setminus (A \cap B)) = (A \cap B) \triangle (A \cap C)$. □

Miscellaneous rules

FOUNDATIONS_03_7383417205293056

Proposition 3.6. Let A, B be classes. Then

$$A \subseteq B \quad \text{iff} \quad A \triangle B = B \setminus A.$$

Proof. Case $A \subseteq B$. Then $A \cup B = B$ and $A \cap B = A$. Hence the thesis. End.

Case $A \triangle B = B \setminus A$. Let $a \in A$. Then $a \notin B \setminus A$. Hence $a \notin A \triangle B$. Thus $a \notin A \cup B$ or $a \in A \cap B$. Indeed $A \triangle B = (A \cup B) \setminus (A \cap B)$. If $a \notin A \cup B$ then we have a contradiction. Therefore $a \in A \cap B$. Then we have the thesis. End. □

FOUNDATIONS_03_4490230937681920

Proposition 3.7. Let A, B, C be classes. Then

$$A \triangle B = A \triangle C \quad \text{iff} \quad B = C.$$

Proof. Case $A \triangle B = A \triangle C$.

Let us show that $B \subseteq C$. Let $b \in B$.

Case $b \in A$. Then $b \notin A \triangle B$. Hence $b \notin A \triangle C$. Therefore $b \in A \cap C$. Indeed $A \triangle C = (A \cup C) \setminus (A \cap C)$. Hence $b \in C$. End.

Case $b \notin A$. Then $b \in A \triangle B$. Indeed $b \in A \cup B$ and $b \notin A \cap B$. Hence $b \in A \triangle C$. Thus $b \in A \cup C$ and $b \notin A \cap C$. Therefore $b \in A$ or $b \in C$. Then we have the thesis. End. End.

Let us show that $C \subseteq B$. Let $c \in C$.

Case $c \in A$. Then $c \notin A \triangle C$. Hence $c \notin A \triangle B$. Therefore $c \in A \cap B$. Indeed $c \notin A \cup B$ or $c \in A \cap B$. Hence $c \in B$. End.

Case $c \notin A$. Then $c \in A \triangle C$. Indeed $c \in A \cup C$ and $c \notin A \cap C$. Hence $c \in A \triangle B$. Thus $c \in A \cup B$ and $c \notin A \cap B$. Therefore $c \in A$ or $c \in B$. Then we have the thesis. End. End. End. □

FOUNDATIONS_03_4578696040022016

Proposition 3.8. Let A be a class. Then

$$A \triangle A = \emptyset.$$

Proof. $A \triangle A = (A \cup A) \setminus (A \cap A) = A \setminus A = \emptyset$. □

FOUNDATIONS_03_6698730398941184

Proposition 3.9. Let A be a class. Then

$$A \triangle \emptyset = A.$$

Proof. $A \triangle \emptyset = (A \cup \emptyset) \setminus (A \cap \emptyset) = A \setminus \emptyset = A$. □

FOUNDATIONS_03_6111806917443584

Proposition 3.10. Let A, B be classes. Then

$$A = B \quad \text{iff} \quad A \triangle B = \emptyset.$$

Proof. Case $A = B$. Then $A \triangle B = (A \cup A) \setminus (A \cap A) = A \setminus A = \emptyset$. Hence the thesis. End.

Case $A \triangle B = \emptyset$. Then $(A \cup B) \setminus (A \cap B)$ is empty. Hence every element of $A \cup B$ is an element of $A \cap B$. Thus for all objects u if $u \in A$ or $u \in B$ then $u \in A$ and $u \in B$. Therefore every element of A is an element of B. Every element of B is an element of A. Then we have the thesis. End. □

Chapter 4

Ordered pairs and Cartesian products

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[readtex `foundations/sections/01_classes.ftl.tex`]

4.1 Pairs

FOUNDATIONS_04_8464577431863296

Axiom 4.1. Let a, a', b, b' be objects. Then

$$(a, b) = (a', b') \quad \text{implies} \quad (a = a' \text{ and } b = b').$$

FOUNDATIONS_04_4782386822774784

Definition 4.2. A pair is an object p such that $p = (a, b)$ for some objects a, b.

Let an ordered pair stand for a pair.

FOUNDATIONS_04_6746145623638016

Definition 4.3. Let p be a pair. $\pi_1 p$ is the object a such that $p = (a, b)$ for some object b.

Let the first entry of p stand for $\pi_1 p$. Let the first component of p stand for the first entry of p.

FOUNDATIONS_04_3750179243032576

Definition 4.4. Let p be a pair. $\pi_2 p$ is the object b such that $p = (a, b)$ for some object a.

Let the second entry of p stand for $\pi_2 p$. Let the second component of p stand for the second entry of p.

4.2 Cartesian products

FOUNDATIONS_04_2877806274936832

Definition 4.5. Let A, B be classes. The Cartesian product of A and B is

$$\{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Let the direct product of A and B stand for the Cartesian product of A and B. Let $A \times B$ stand for the Cartesian product of A and B.

FOUNDATIONS_04_1581118511906816

Proposition 4.6. Let A, B be classes and a, b be objects. Then

$$(a, b) \in A \times B \quad \text{iff} \quad (a \in A \text{ and } b \in B).$$

Proof. Case $(a, b) \in A \times B$. We can take $a' \in A$ and $b' \in B$ such that $(a, b) = (a', b')$. Then $a = a'$ and $b = b'$. Hence $a \in A$ and $b \in B$. End.

Case $a \in A$ and $b \in B$. a and a are objects. Hence (a, b) is an object. Therefore $(a, b) \in A \times B$. End. □

FOUNDATIONS_04_2198552029691904

Proposition 4.7. Let A, B be classes. Then $A \times B$ is empty iff A is empty or B is empty.

Proof. Case $A \times B$ is empty. Assume that A and B are nonempty. Then we can take an element a of A and an element b of B. Then $(a, b) \in A \times B$. Contradiction. End.

Case A is empty or B is empty. Assume that $A \times B$ is nonempty. Then we can take an element c of $A \times B$. Then $c = (a, b)$ for some $a \in A$ and some $b \in B$. Hence A and B are nonempty. Contradiction. End. □

FOUNDATIONS_04_7971087096741888

Proposition 4.8. Let a, b be objects. Then

$$\{a\} \times \{b\} = \{(a, b)\}.$$

Proof. Let us show that $\{a\} \times \{b\} \subseteq \{(a, b)\}$. Let $c \in \{a\} \times \{b\}$. Take $a' \in \{a\}$ and $b' \in \{b\}$ such that $c = (a', b')$. We have $a' = a$ and $b' = b$. Hence $c = (a, b)$. Thus $c \in \{(a, b)\}$. End.

Let us show that $\{(a, b)\} \subseteq \{a\} \times \{b\}$. Let $c \in \{(a, b)\}$. Then $c = (a, b)$. We have $a \in \{a\}$ and $b \in \{b\}$. Hence $c \in \{a\} \times \{b\}$. End. □

FOUNDATIONS_04_7456594440749056

Proposition 4.9. Let A, A', B, B' be nonempty classes. Then

$$A \times B = A' \times B' \quad \text{implies} \quad (A = A' \text{ and } B = B').$$

Proof. Assume $A \times B = A' \times B'$.

(1) $A \subseteq A'$ and $B \subseteq B'$.
Proof. Let $a \in A$ and $b \in B$. Then $(a, b) \in A \times B$. Hence $(a, b) \in A' \times B'$. Thus $a \in A'$ and $b \in B'$. Qed.

(2) $A' \subseteq A$ and $B' \subseteq B$.
Proof. Let $a \in A'$ and $b \in B'$. Then $(a, b) \in A' \times B'$. Hence $(a, b) \in A \times B$. Thus $a \in A$ and $b \in B$. Qed. □

Chapter 5

Computation laws for Cartesian products

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[readtex `foundations/sections/04_pairs-and-products.ftl.tex`]

Subclass laws

FOUNDATIONS_05_5719644021194752

Proposition 5.1. Let A, B, C be classes. Then

$$A \subseteq B \quad \text{implies} \quad A \times C \subseteq B \times C.$$

Proof. Assume $A \subseteq B$. Let $x \in A \times C$. Take $a \in A$ and $c \in C$ such that $x = (a, c)$. Then $a \in B$. Hence $(a, c) \in B \times C$. □

FOUNDATIONS_05_4888282951319552

Proposition 5.2. Let A, A', B, B' be classes. Assume that A and A' are nonempty. Then

$$(A \times A') \subseteq (B \times B') \quad \text{iff} \quad (A \subseteq B \text{ and } A' \subseteq B').$$

Proof. Case $(A \times A') \subseteq (B \times B')$. Let us show that for all $a \in A$ and all $a' \in A'$ we have $a \in B$ and $a' \in B'$. Let $a \in A$ and $a' \in A'$. Then $(a, a') \in A \times A'$. Hence $(a, a') \in B \times B'$. Thus $a \in B$ and $a' \in B'$. End. End.

Case $A \subseteq B$ and $A' \subseteq B'$. Let $x \in A \times A'$. Take $a \in A$ and $a' \in A'$ such that $x = (a, a')$. Then $a \in B$ and $a' \in B'$. Hence $(a, a') \in B \times B'$. End. □

Distributivity of product and union

FOUNDATIONS_05_8849658323402752

Proposition 5.3. Let A, B, C be classes. Then

$$(A \cup B) \times C = (A \times C) \cup (B \times C).$$

Proof. Let us show that $((A \cup B) \times C) \subseteq (A \times C) \cup (B \times C)$. Let $x \in (A \cup B) \times C$. Take $y \in A \cup B$ and $c \in C$ such that $x = (y, c)$. Then $y \in A$ or $y \in B$. If $y \in A$ then $x \in A \times C$ and if $y \in B$ then $x \in B \times C$. Hence $x \in A \times C$ or $x \in B \times C$. Thus $x \in (A \times C) \cup (B \times C)$. End.

Let us show that $((A \times C) \cup (B \times C)) \subseteq (A \cup B) \times C$. Let $x \in (A \times C) \cup (B \times C)$. Then $x \in A \times C$ or $x \in B \times C$. Take objects y, c such that $x = (y, c)$. Then ($y \in A$ or $y \in B$) and $c \in C$. Hence $y \in A \cup B$. Thus $x \in (A \cup B) \times C$. End. □

FOUNDATIONS_05_476526841692160

Proposition 5.4. Let A, B, C be classes. Then

$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

Proof. Let us show that $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$. Let $x \in A \times (B \cup C)$. Take $a \in A$ and $y \in B \cup C$ such that $x = (a, y)$. Then $y \in B$ or $y \in C$. Hence $x \in A \times B$ or $x \in A \times C$. Indeed if $y \in B$ then $x \in A \times B$ and if $y \in C$ then $x \in A \times C$. Thus $x \in (A \times B) \cup (A \times C)$. End.

Let us show that $((A \times B) \cup (A \times C)) \subseteq A \times (B \cup C)$. Let $x \in (A \times B) \cup (A \times C)$. Then $x \in A \times B$ or $x \in A \times C$. Take objects a, y such that $x = (a, y)$. Then $a \in A$ and ($y \in B$ or $y \in C$). Hence $x \in A \times (B \cup C)$. End. □

Distributivity of product and intersection

FOUNDATIONS_05_1249567930580992

Proposition 5.5. Let A, B, C be classes. Then

$$(A \cap B) \times C = (A \times C) \cap (B \times C).$$

Proof. Let us show that $((A \cap B) \times C) \subseteq (A \times C) \cap (B \times C)$. Let $x \in (A \cap B) \times C$. Take $y \in A \cap B$ and $c \in C$ such that $x = (y, c)$. Then $y \in A$ and $y \in B$. Hence $x \in A \times C$ and $x \in B \times C$. Thus $x \in (A \times C) \cap (B \times C)$. End.

Let us show that $((A \times C) \cap (B \times C)) \subseteq (A \cap B) \times C$. Let $x \in (A \times C) \cap (B \times C)$. Then $x \in A \times C$ and $x \in B \times C$. Take objects y, z such that $x = (y, z)$. Then ($y \in A$ and $y \in B$) and $z \in C$. Hence $y \in A \cap B$. Thus $x \in (A \cap B) \times C$. End. ☐

FOUNDATIONS_05_954964241285120

Proposition 5.6. Let A, B, C be classes. Then

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

Proof. Let us show that $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$. Let $x \in A \times (B \cap C)$. Take $a \in A$ and $b \in B \cap C$ such that $x = (a, b)$. Then $b \in B$ and $b \in C$. Hence $x \in A \times B$ and $x \in A \times C$. Thus $x \in (A \times B) \cap (A \times C)$. End.

Let us show that $((A \times B) \cap (A \times C)) \subseteq A \times (B \cap C)$. Let $x \in (A \times B) \cap (A \times C)$. Then $x \in A \times B$ and $x \in A \times C$. Take objects y, z such that $x = (y, z)$. Then $y \in A$ and ($z \in B$ and $z \in C$). Hence $x \in A \times (B \cap C)$. End. ☐

Distributivity of product and complement

FOUNDATIONS_05_6495329908162560

Proposition 5.7. Let A, B, C be classes. Then

$$(A \setminus B) \times C = (A \times C) \setminus (B \times C).$$

Proof. Let us show that $((A \setminus B) \times C) \subseteq (A \times C) \setminus (B \times C)$. Let $x \in (A \setminus B) \times C$. Take $a \in A \setminus B$ and $c \in C$ such that $x = (a, c)$. Then $a \in A$ and $a \notin B$. Hence $x \in A \times C$ and $x \notin B \times C$. Thus $x \in (A \times C) \setminus (B \times C)$. End.

Let us show that $((A \times C) \setminus (B \times C)) \subseteq (A \setminus B) \times C$. Let $x \in (A \times C) \setminus (B \times C)$. Then $x \in A \times C$ and $x \notin B \times C$. Take $a \in A$ and $c \in C$ such that $x = (a, c)$. Then $a \notin B$.

Indeed if $a \in B$ then $x \in B \times C$. Hence $a \in A \setminus B$. Thus $x \in (A \setminus B) \times C$. End. □

FOUNDATIONS_05_3195639422779392

Proposition 5.8. Let A, B, C be classes. Then

$$A \times (B \setminus C) = (A \times B) \setminus (A \times C).$$

Proof. Let us show that $A \times (B \setminus C) \subseteq (A \times B) \setminus (A \times C)$. Let $x \in A \times (B \setminus C)$. Take $a \in A$ and $b \in B \setminus C$ such that $x = (a, b)$. Then $b \in B$ and $b \notin C$. Hence $x \in A \times B$ and $x \notin A \times C$. Thus $x \in (A \times B) \setminus (A \times C)$. End.

Let us show that $((A \times B) \setminus (A \times C)) \subseteq A \times (B \setminus C)$. Let $x \in (A \times B) \setminus (A \times C)$. Then $x \in A \times B$ and $x \notin A \times C$. Take objects a, b such that $x = (a, b)$. Then $a \in A$ and ($b \in B$ and $b \notin C$). Hence $x \in A \times (B \setminus C)$. End. □

Equality law

FOUNDATIONS_05_2677218429894656

Proposition 5.9. Let A, A', B, B' be classes. Assume that A and A' are nonempty or B and B' are nonempty. Then

$$(A \times A') = (B \times B') \quad \text{iff} \quad (A = B \text{ and } A' = B').$$

Proof. Case $A \times A' = B \times B'$. Then A and A' are nonempty iff B and B' are nonempty.

Let us show that for all $a \in A$ and all $a' \in A'$ we have $a \in B$ and $a' \in B'$. Let $a \in A$ and $a' \in A'$. Then $(a, a') \in A \times A'$. Hence we can take $x \in B \times B'$ such that $x = (a, a')$. Thus $a \in B$ and $a' \in B'$. End.

Therefore $A \subseteq B$ and $A' \subseteq B'$. Indeed A and A' are nonempty.

Let us show that for all $b \in B$ and all $b' \in B'$ we have $b \in A$ and $b' \in A'$. Let $b \in B$ and $b' \in B'$. Then $(b, b') \in B \times B'$. Hence we can take $x \in A \times A'$ such that $x = (b, b')$. Thus $(b, b') \in A \times A'$. End.

Therefore $B \subseteq A$ and $B' \subseteq A'$. Indeed B and B' are nonempty. End.

Case $A = B$ and $A' = B'$. Trivial. □

Intersections of products

FOUNDATIONS_05_4154592050806784

Proposition 5.10. Let A, A', B, B' be classes. Then

$$(A \times B) \cap (A' \times B') = (A \cap A') \times (B \cap B').$$

Proof. Let us show that $((A \times B) \cap (A' \times B')) \subseteq (A \cap A') \times (B \cap B')$. Let $x \in (A \times B) \cap (A' \times B')$. Then $x \in A \times B$ and $x \in A' \times B'$. Take objects a, b such that $x = (a, b)$. Then $a \in A, A'$ and $b \in B, B'$. Hence $a \in A \cap A'$ and $b \in B \cap B'$. Thus $x \in (A \cap A') \times (B \cap B')$. End.

Let us show that $(A \cap A') \times (B \cap B') \subseteq (A \times B) \cap (A' \times B')$. Let $x \in (A \cap A') \times (B \cap B')$. Take elements a, b such that $x = (a, b)$. Then $a \in A \cap A'$ and $b \in B \cap B'$. Hence $a \in A, A'$ and $b \in B, B'$. Thus $x \in A \times B$ and $x \in A' \times B'$. Therefore $x \in (A \times B) \cap (A' \times B')$. End. □

Unions of products

FOUNDATIONS_05_7090174334861312

Proposition 5.11. Let A, A', B, B' be classes. Then

$$(A \times B) \cup (A' \times B') \subseteq (A \cup A') \times (B \cup B').$$

Proof. Let $x \in (A \times B) \cup (A' \times B')$. Then $x \in A \times B$ or $x \in A' \times B'$. Take objects a, b such that $x = (a, b)$. Then ($a \in A$ or $a \in A'$) and ($b \in B$ or $b \in B'$). Hence $a \in A \cup A'$ and $b \in B \cup B'$. Thus $x \in (A \cup A') \times (B \cup B')$. □

Complements of products

FOUNDATIONS_05_5552125989879808

Proposition 5.12. Let A, A', B, B' be classes. Then

$$(A \times B) \setminus (A' \times B') = (A \times (B \setminus B')) \cup ((A \setminus A') \times B).$$

Proof. Let us show that $((A \times B) \setminus (A' \times B')) \subseteq (A \times (B \setminus B')) \cup ((A \setminus A') \times B)$. Let $x \in (A \times B) \setminus (A' \times B')$. Then $x \in A \times B$ and $x \notin A' \times B'$. Take $a \in A$ and $b \in B$ such that $x = (a, b)$. Then it is wrong that $a \in A'$ and $b \in B'$. Hence $a \notin A'$ or $b \notin B'$. Thus $a \in A \setminus A'$ or $b \in B \setminus B'$. Therefore $x \in A \times (B \setminus B')$ or $x \in (A \setminus A') \times B$. Hence we have $x \in (A \times (B \setminus B')) \cup ((A \setminus A') \times B)$. End.

Let us show that $(A \times (B \setminus B')) \cup ((A \setminus A') \times B) \subseteq (A \times B) \setminus (A' \times B')$. Let $x \in (A \times (B \setminus B')) \cup ((A \setminus A') \times B)$. Then $x \in (A \times (B \setminus B'))$ or $x \in ((A \setminus A') \times B)$. Take elements a, b such that $x = (a, b)$. Then ($a \in A$ and $b \in B \setminus B'$) or ($a \in A \setminus A'$ and $b \in B$).

Case $a \in A$ and $b \in B \setminus B'$. Then $a \in A$ and $b \in B$. Hence $x \in A \times B$. We have $b \notin B'$. Thus $x \notin A' \times B'$. Therefore $x \in (A \times B) \setminus (A' \times B')$. End.

Case $a \in A \setminus A'$ and $b \in B$. Then $a \in A$ and $b \in B$. Hence $x \in A \times B$. We have $a \notin A'$. Thus $x \notin A' \times B'$. Therefore $x \in (A \times B) \setminus (A' \times B')$. End. End. □

Chapter 6

Maps

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[readtex `foundations/sections/04_pairs-and-products.ftl.tex`]

6.1 Ranges

FOUNDATIONS_06_4284980337311744

Definition 6.1. Let f be a map. A value of f is an object b such that $b = f(a)$ for some $a \in \mathrm{dom}(f)$.

FOUNDATIONS_06_1938831225913344

Definition 6.2. Let f be a map. The range of f is

$$\{f(a) \mid a \in \mathrm{dom}(f)\}.$$

Let $\mathrm{range}(f)$ stand for the range of f.

FOUNDATIONS_06_6386349418479616

Proposition 6.3. Let f be a map and b be an object. b is a value of f iff $b \in \mathrm{range}(f)$.

Proof. Case b is a value of f. Take $a \in \mathrm{dom}(f)$ such that $b = f(a)$. b is an object. Hence $b \in \mathrm{range}(f)$. End.

Case $b \in \mathrm{range}(f)$. Then b is an object such that $b = f(a)$ for some $a \in \mathrm{dom}(f)$. Hence b is a value of f. End. □

6.2 The identity map

FOUNDATIONS_06_1920902360989696

Definition 6.4. Let A be a class. id_A is the map h such that h is defined on A and $h(a) = a$ for all $a \in A$.

Let the identity map on A stand for id_A.

6.3 Composition

FOUNDATIONS_06_7605717729017856

Definition 6.5. Let f, g be maps. Assume $\mathrm{range}(f) \subseteq \mathrm{dom}(g)$. $g \circ f$ is the map h such that h is defined on $\mathrm{dom}(f)$ and $h(a) = g(f(a))$ for all $a \in \mathrm{dom}(f)$.

Let the composition of g and f stand for $g \circ f$.

6.4 Restriction

FOUNDATIONS_06_7095412741636096

Definition 6.6. Let f be a map and $X \subseteq \mathrm{dom}(f)$. $f \upharpoonright X$ is the map h such that h is defined on X and $h(a) = f(a)$ for all $a \in X$.

Let the restriction of f to X stand for $f \upharpoonright X$.

FOUNDATIONS_06_2170189258948608

Proposition 6.7. Let A be a class and $X \subseteq A$. Then $\mathrm{id}_A \upharpoonright X = \mathrm{id}_X$.

6.5 Images and preimages

FOUNDATIONS_06_3038237683613696

Definition 6.8. Let f be a map and A be a class. The image of A under f is

$$\{f(a) \mid a \in \mathrm{dom}(f) \cap A\}.$$

Let the direct image of A under f stand for the image of A under f. Let $f_*(A)$ stand for the image of A under f.

Let $f[A]$ stand for $f_*(A)$.

FOUNDATIONS_06_4563167805964288

Definition 6.9. Let f be a map and B be a class. The preimage of B under f is

$$\{a \in \mathrm{dom}(f) \mid f(a) \in B\}.$$

Let the inverse image of B under f stand for the preimage of B under f. Let $f^*(B)$ stand for the preimage of B under f.

6.6 Maps between classes

FOUNDATIONS_06_6934038600220672

Definition 6.10. Let A be a class. A map of A is a map f such that $\mathrm{dom}(f) = A$.

FOUNDATIONS_06_7725375157174272

Definition 6.11. Let B be a class. A map to B is a map f such that $f(a) \in B$ for each $a \in \mathrm{dom}(f)$.

FOUNDATIONS_06_2823507398361088

Definition 6.12. Let A, B be classes. A map from A to B is a map f such that $\mathrm{dom}(f) = A$ and $f(a) \in B$ for each $a \in A$.

Let $f : A \to B$ stand for f is a map from A to B.

FOUNDATIONS_06_3390734908522496

Definition 6.13. Let A be a class. A map on A is a map from A to A.

FOUNDATIONS_06_3312973569327104

Proposition 6.14. Let A, B be classes and $f, g : A \to B$. Assume that $f(a) = g(a)$ for all $a \in A$. Then $f = g$.

Proposition 6.15. Let A, B be classes and f be a map of A. Assume that $f(a) \in B$ for all $a \in A$. Then f is a map from A to B iff $\mathrm{range}(f) \subseteq B$.

FOUNDATIONS_06_5104361690628096

Proposition 6.16. Let A be a class. Then id_A is a map on A.

FOUNDATIONS_06_1706446651654144

Proposition 6.17. Let A, B, C be classes and $f : A \to B$ and $g : B \to C$. Then $g \circ f : A \to C$.

FOUNDATIONS_06_4078561256275968

Proposition 6.18. Let A, B be classes and $f : A \to B$ and $X \subseteq A$. Then $f \restriction X : X \to B$.

FOUNDATIONS_06_3964401904254976

Proposition 6.19. Let A, B be classes and $f : A \to B$. Then

$$f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f.$$

Proof. A is the domain of $f \circ \mathrm{id}_A$ and the domain of f and the domain of $\mathrm{id}_B \circ f$. We have $(f \circ \mathrm{id}_A)(a) = f(\mathrm{id}_A(a)) = f(a) = \mathrm{id}_B(f(a)) = (\mathrm{id}_B \circ f)(a)$ for all $a \in A$. Hence

$f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$. □

FOUNDATIONS_06_3118771061391360

Proposition 6.20. Let A be a class and $X \subseteq A$. Then

$$\mathrm{id}_A \upharpoonright X = \mathrm{id}_X .$$

Proof. We have $\mathrm{dom}(\mathrm{id}_A \upharpoonright X) = X = \mathrm{dom}(\mathrm{id}_X)$. $(\mathrm{id}_A \upharpoonright X)(a) = \mathrm{id}_A(a) = a = \mathrm{id}_X(a)$ for all $a \in X$. Hence $\mathrm{id}_A \upharpoonright X = \mathrm{id}_X$. □

FOUNDATIONS_06_6866147389472768

Proposition 6.21. Let A, B, C, D be classes and $f : A \to B$ and $g : B \to C$ and $h : C \to D$. Then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Proof. $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are maps from A to D.

Let us show that $(h \circ (g \circ f))(a) = ((h \circ g) \circ f)(a)$ for all $a \in A$. Let $a \in A$. Then $(h \circ (g \circ f))(a) = h((g \circ f)(a)) = h(g(f(a))) = (h \circ g)(f(a)) = ((h \circ g) \circ f)(a)$. End.

Hence $h \circ (g \circ f) = (h \circ g) \circ f$. □

6.7 Maps and products

FOUNDATIONS_06_5135634870042624

Definition 6.22. Let f be a map such that $\mathrm{dom}(f) = A \times B$ for some nonempty classes A, B. Let a be an object such that $(a, b) \in \mathrm{dom}(f)$ for some object b. $f(a, -)$ is the map such that $\mathrm{dom}(f(a, -)) = B$ and $f(a, -)(b) = f(a, b)$ for all $b \in B$ where B is the class such that $\mathrm{dom}(f) = A \times B$ for some class A.

FOUNDATIONS_06_3621991366000640

Definition 6.23. Let f be a map such that $\mathrm{dom}(f) = A \times B$ for some nonempty classes A, B. Let b be an object such that $(a, b) \in \mathrm{dom}(f)$ for some object a. $f(-, b)$ is the map such that $\mathrm{dom}(f(-, b)) = A$ and $f(-, b)(a) = f(a, b)$ for all $a \in A$ where A is the class such that $\mathrm{dom}(f) = A \times B$ for some class B.

FOUNDATIONS_06_8946256734846976

Proposition 6.24. Let A, B, C be classes such that A, B are nonempty and $a \in A$. Let f be a map from $A \times B$ to C. Then $f(a, -)$ is a map from B to C.

FOUNDATIONS_06_8080207992848384

Proposition 6.25. Let A, B, C be classes such that A, B are nonempty and $b \in B$. Let f be a map from $A \times B$ to C. Then $f(-, b)$ is a map from A to C.

FOUNDATIONS_06_2754759509409792

Proposition 6.26. Let A, B, C be classes and f be a map of $A \times B$. Assume that $f(a, b) \in C$ for all $a \in A$ and all $b \in B$. Then f is a map from $A \times B$ to C.

FOUNDATIONS_06_2304295212941312

Proposition 6.27. Let A, B, C be classes and f be a map from $A \times B$ to C. Let $a \in A$ and $b \in B$. Then $f(a, b) \in C$.

Chapter 7

Computation laws for images and preimages

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[readtex foundations/sections/06_maps.ftl.tex]

FOUNDATIONS_07_5919649206108160

Proposition 7.1. Let A, B be classes and $f : A \to B$ and $X \subseteq A$. Then

$$f_*(X) = \{f(x) \mid x \in X\}.$$

FOUNDATIONS_07_5543924730953728

Corollary 7.2. Let A, B be classes and $f : A \to B$. Then

$$f_*(A) = \mathrm{range}(f).$$

FOUNDATIONS_07_1818812171157504

Corollary 7.3. Let A, B be classes and $f : A \to B$ and $X \subseteq A$. Then

$$f_*(X) = \text{range}(f \upharpoonright X).$$

FOUNDATIONS_07_911395830890496

Proposition 7.4. Let A be a class and $X \subseteq A$. Then

$$(\text{id}_A)_*(X) = X.$$

FOUNDATIONS_07_3349817830932480

Proposition 7.5. Let B be a class and $Y \subseteq B$. Then

$$(\text{id}_B)^*(Y) = Y.$$

FOUNDATIONS_07_6362984433582080

Proposition 7.6. Let A, B be classes and $f : A \to B$ and $Y \subseteq B$ and $a \in A$. Then

$$f(a) \in Y \quad \text{iff} \quad a \in f^*(Y).$$

Proof. We have $f^*(Y) = \{x \in A \mid f(x) \in Y\}$. Hence $a \in f^*(Y)$ iff $a \in A$ and $f(a) \in Y$. Then we have the thesis. □

FOUNDATIONS_07_6730546254184448

Proposition 7.7. Let A, B be classes and $f : A \to B$. Then

$$f_*(A) \subseteq B.$$

Proof. $f_*(A) = f_*(\text{dom}(f)) = \text{range}(f) \subseteq B$. □

FOUNDATIONS_07_6541963008409600

Proposition 7.8. Let A, B be classes and $f : A \to B$. Then

$$f^*(B) = A.$$

Proof. We have $f^*(B) = \{a \in A \mid f(a) \in B\}$. $f(a) \in B$ for all $a \in A$. Hence the thesis. □

FOUNDATIONS_07_1913313581596672

Proposition 7.9. Let A, B be classes and $f : A \to B$. Then

$$f_*(f^*(B)) = f_*(A).$$

Proof. Let us show that $f_*(f^*(B)) \subseteq f_*(A)$. Let $b \in f_*(f^*(B))$. Take $a \in f^*(B) \cap A$ such that $b = f(a)$. Then $a \in A$. Hence $b \in f_*(A)$. End.

Let us show that $f_*(A) \subseteq f_*(f^*(B))$. Let $b \in f_*(A)$. Take $a \in A$ such that $b = f(a)$. We have $b \in B$. Hence $a \in f^*(B)$. Thus $f(a) \in f_*(f^*(B))$. Indeed $f^*(B) \subseteq A$. Therefore $b \in f_*(f^*(B))$. End. □

FOUNDATIONS_07_3819758101200896

Proposition 7.10. Let A, B be classes and $f : A \to B$. Then

$$f^*(f_*(A)) = A.$$

Proof. $f^*(f_*(A)) = \{a \in A \mid f(a) \in f_*(A)\}$. For all $a \in A$ we have $f(a) \in f_*(A)$. Hence every element of $f^*(f_*(A))$ is contained in A and every element of A is contained in $f^*(f_*(A))$. Thus $f^*(f_*(A)) = A$. □

FOUNDATIONS_07_7760514696347648

Proposition 7.11. Let A, B be classes and $f : A \to B$ and $Y \subseteq B$. Then

$$f_*(f^*(Y)) = Y \cap f_*(A).$$

Proof. Let us show that $f_*(f^*(Y)) \subseteq Y \cap f_*(A)$. Let $b \in f_*(f^*(Y))$. Take $a \in f^*(Y)$ such that $b = f(a)$. Then $f(a) \in Y \cap f_*(A)$. Hence we have $b \in Y \cap f_*(A)$. End.

Let us show that $Y \cap f_*(A) \subseteq f_*(f^*(Y))$. Let $b \in Y \cap f_*(A)$. Take $a \in A$ such that $b = f(a)$. Then $a \in f^*(Y)$. Hence $f(a) \in f_*(f^*(Y))$. End. □

FOUNDATIONS_07_5585105345052672

Corollary 7.12. Let A, B be classes and $f : A \to B$ and $Y \subseteq B$. Then

$$f_*(f^*(Y)) \subseteq Y.$$

FOUNDATIONS_07_4890896170483712

Proposition 7.13. Let A, B be classes and $f : A \to B$ and $X \subseteq A$. Then

$$f^*(f_*(X)) \supseteq X.$$

Proof. Let $a \in X$. Then $f(a) \in f_*(X)$. Hence $a \in f^*(f_*(X))$. Indeed $f_*(X) \subseteq B$. □

FOUNDATIONS_07_3318372355801088

Proposition 7.14. Let A, B be classes and $f : A \to B$ and $X \subseteq A$. Then

$$f_*(X) = \emptyset \quad \text{iff} \quad X = \emptyset.$$

Proof. Case $f_*(X)$ is empty. Then there is no $a \in X$ such that $f(a) \in f_*(X)$. For all $a \in X$ we have $f(a) \in f_*(X)$. Hence X is empty. End.

Case X is empty. For all $b \in f_*(X)$ we have $b = f(a)$ for some $a \in X$. There is no $a \in X$. Hence $f_*(X)$ is empty. End. □

FOUNDATIONS_07_8597874786959360

Proposition 7.15. Let A, B be classes and $f : A \to B$ and $Y \subseteq B$. Then

$$f^*(Y) = \emptyset \quad \text{iff} \quad Y \subseteq B \setminus f_*(A).$$

Proof. Case $f^*(Y)$ is empty. Let $b \in Y$. Then $b \in B$.

There is no $a \in A$ such that $b = f(a)$.
Proof. Assume the contrary. Take $a \in A$ such that $b = f(a)$. Then $a \in f^*(Y)$. Contradiction. Qed.

Hence $b \notin f_*(A)$. Therefore $b \in B \setminus f_*(A)$. End.

Case $Y \subseteq B \setminus f_*(A)$. Then no element of Y is an element of $f_*(A)$. Assume that $f^*(Y)$ is nonempty. Take $a \in f^*(Y)$. Then $f(a) \in Y$ and $f(a) \in f_*(A)$. Contradiction. End. □

FOUNDATIONS_07_6295504988143616

Proposition 7.16. Let A, B be classes and $f : A \to B$ and $X \subseteq A$ and $Y \subseteq B$. Then

$$f_*(X) \cap Y = \emptyset \quad \text{iff} \quad X \cap f^*(Y) = \emptyset.$$

Proof. Case $f_*(X) \cap Y$ is empty. Assume that $X \cap f^*(Y)$ is nonempty. Take $a \in X \cap f^*(Y)$. Then $f(a) \in f_*(X)$ and $f(a) \in Y$. Hence $f(a) \in f_*(X) \cap Y$. Contradiction. End.

Case $X \cap f^*(Y)$ is empty. Assume that $f_*(X) \cap Y$ is nonempty. Take $b \in f_*(X) \cap Y$. Consider a $a \in X$ such that $b = f(a)$. Then $a \in f^*(Y)$. Indeed $b \in Y$. Hence $a \in X \cap f^*(Y)$. Contradiction. End. □

FOUNDATIONS_07_5628919411638272

Proposition 7.17. Let A, B, C be classes and $f : A \to B$ and $g : B \to C$ and $X \subseteq A$. Then

$$(g \circ f)_*(X) = g_*(f_*(X)).$$

Proof. $((g \circ f)_*(X)) = \{g(f(a)) \mid a \in X\}$. $f_*(X)$ is a subclass of B. We have $g_*(f_*(X)) = \{g(b) \mid b \in f_*(X)\}$ and $f_*(X) = \{f(a) \mid a \in X\}$. Thus $g_*(f_*(X)) = \{g(f(a)) \mid a \in X\}$. Therefore $(g \circ f)_*(X) = g_*(f_*(X))$. □

FOUNDATIONS_07_6824917886566400

Proposition 7.18. Let A, B, C be classes and $f : A \to B$ and $g : B \to C$ and $Z \subseteq C$. Then

$$(g \circ f)^*(Z) = f^*(g^*(Z)).$$

Proof. $((g \circ f)^*(Z)) = \{a \in A \mid g(f(a)) \in Z\}$. We have $g^*(Z) = \{b \in B \mid g(b) \in Z\}$ and $f^*(g^*(Z)) = \{a \in A \mid f(a) \in g^*(Z)\}$. Hence $f^*(g^*(Z)) = \{a \in A \mid g(f(a)) \in Z\}$. Thus $(g \circ f)^*(Z) = f^*(g^*(Z))$. □

FOUNDATIONS_07_7396318576115712

Proposition 7.19. Let A, B be classes and $f : A \to B$ and $X, X' \subseteq A$. Then

$$X \subseteq X' \quad \text{implies} \quad f_*(X) \subseteq f_*(X').$$

Proof. Assume $X \subseteq X'$. Let $b \in f_*(X)$. Take $a \in X$ such that $f(a) = b$. Then $a \in X'$. Hence $b = f(a) \in f_*(X')$. □

FOUNDATIONS_07_8376448628817920

Proposition 7.20. Let A, B be classes and $f : A \to B$ and $Y, Y' \subseteq B$. Then

$$Y \subseteq Y' \quad \text{implies} \quad f^*(Y) \subseteq f^*(Y').$$

Proof. Assume $Y \subseteq Y'$. Let $a \in f^*(Y)$. Then $f(a) \in Y$. Hence $f(a) \in Y'$. Thus $a \in f^*(Y')$. □

FOUNDATIONS_07_4448961469349888

Proposition 7.21. Let A, B be classes and $f : A \to B$ and $X, X' \subseteq A$. Then

$$f_*(X \cup X') = f_*(X) \cup f_*(X').$$

Proof. Let us show that $f_*(X \cup X') \subseteq f_*(X) \cup f_*(X')$. Let $b \in f_*(X \cup X')$. Take $a \in X \cup X'$ such that $b = f(a)$. Then $a \in X$ or $a \in X'$. Hence $f(a) \in f_*(X)$ or $f(a) \in f_*(X')$. Thus $b = f(a) \in f_*(X) \cup f_*(X')$. End.

Let us show that $f_*(X) \cup f_*(X') \subseteq f_*(X \cup X')$. Let $b \in f_*(X) \cup f_*(X')$.

Case $b \in f_*(X)$. Take $a \in X$ such that $b = f(a)$. Then $a \in X \cup X'$. Hence $b \in f_*(X \cup X')$. End.

Case $b \in f_*(X')$. Take $a \in X'$ such that $b = f(a)$. Then $a \in X \cup X'$. Hence $b \in f_*(X \cup X')$. End. End. □

FOUNDATIONS_07_1547089051910144

Proposition 7.22. Let A, B be classes and $f : A \to B$ and $Y, Y' \subseteq B$. Then

$$f^*(Y \cup Y') = f^*(Y) \cup f^*(Y').$$

Proof. Let us show that $f^*(Y \cup Y') \subseteq f^*(Y) \cup f^*(Y')$. Let $a \in f^*(Y \cup Y')$. Then $f(a) \in Y \cup Y'$. Hence $f(a) \in Y$ or $f(a) \in Y'$. If $f(a) \in Y$ then $a \in f^*(Y)$. If $f(a) \in Y'$ then $a \in f^*(Y')$. Thus $a \in f^*(Y) \cup f^*(Y')$. End.

Let us show that $f^*(Y) \cup f^*(Y') \subseteq f^*(Y \cup Y')$. Let $a \in f^*(Y) \cup f^*(Y')$. Then $a \in f^*(Y)$ or $a \in f^*(Y')$. If $a \in f^*(Y)$ then $f(a) \in Y$. If $a \in f^*(Y')$ then $f(a) \in Y'$. Hence $f(a) \in Y \cup Y'$. Thus $a \in f^*(Y \cup Y')$. End. □

FOUNDATIONS_07_3966130473402368

Proposition 7.23. Let A, B be classes and $f : A \to B$ and $X, X' \subseteq A$. Then

$$f_*(X \cap X') \subseteq f_*(X) \cap f_*(X').$$

Proof. Let $b \in f_*(X \cap X')$. Take $a \in X \cap X'$ such that $b = f(a)$. Then $a \in X$ and $a \in X'$. Hence $f(a) \in f_*(X)$ and $f(a) \in f_*(X')$. Thus $b \in f_*(X) \cap f_*(X)$. □

FOUNDATIONS_07_4021844428455936

Proposition 7.24. Let A, B be classes and $f : A \to B$ and $Y, Y' \subseteq B$. Then

$$f^*(Y \cap Y') = f^*(Y) \cap f^*(Y').$$

Proof. Let us show that $f^*(Y \cap Y') \subseteq f^*(Y) \cap f^*(Y')$. Let $a \in f^*(Y \cap Y')$. Then $f(a) \in Y \cap Y'$. Hence $f(a) \in Y$ and $f(a) \in Y'$. Thus $a \in f^*(Y)$ and $a \in f^*(Y')$. Therefore $a \in f^*(Y) \cap f^*(Y')$. End.

Let us show that $f^*(Y) \cap f^*(Y') \subseteq f^*(Y \cap Y')$. Let $a \in f^*(Y) \cap f^*(Y')$. Then $a \in f^*(Y)$ and $a \in f^*(Y')$. Hence $f(a) \in Y$ and $f(a) \in Y'$. Thus $f(a) \in Y \cap Y'$. Therefore $a \in f^*(Y \cap Y')$. End. □

FOUNDATIONS_07_8372256617005056

Proposition 7.25. Let A, B be classes and $f : A \to B$ and $X, X' \subseteq A$. Then

$$f_*(X \setminus X') \supseteq f_*(X) \setminus f_*(X').$$

Proof. Let $b \in f_*(X) \setminus f_*(X')$. Then $b \in f_*(X)$ and $b \notin f_*(X')$. Take $a \in X$ such that $b = f(a)$. If $a \in X'$ then $b \in f_*(X')$. Hence $a \notin X'$. Thus $a \in X \setminus X'$. Therefore $b = f(a) \in f_*(X \setminus X')$. □

FOUNDATIONS_07_6552168641331200

Proposition 7.26. Let A, B be classes and $f : A \to B$ and $Y, Y' \subseteq B$. Then

$$f^*(Y \setminus Y') = f^*(Y) \setminus f^*(Y').$$

Proof. Let us show that $f^*(Y \setminus Y') \subseteq f^*(Y) \setminus f^*(Y')$. Let $a \in f^*(Y \setminus Y')$. Then $f(a) \in Y \setminus Y'$. Hence $f(a) \in Y$ and $f(a) \notin Y'$. Thus $a \in f^*(Y)$ and $a \notin f^*(Y')$. Therefore $a \in f^*(Y) \setminus f^*(Y')$. End.

Let us show that $f^*(Y) \setminus f^*(Y') \subseteq f^*(Y \setminus Y')$. Let $a \in f^*(Y) \setminus f^*(Y')$. Then $a \in f^*(Y)$ and $a \notin f^*(Y')$. Hence $f(a) \in Y$ and $f(a) \notin Y'$. Thus $f(a) \in Y \setminus Y'$. Therefore $a \in f^*(Y \setminus Y')$. End. $\square$

Chapter 8

Surjections, injections and bijections

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[readtex `foundations/sections/06_maps.ftl.tex`]

8.1 Surjective maps

FOUNDATIONS_08_8681187805495296

Definition 8.1. Let f be a map and B be a class. f is surjective onto B iff $\mathrm{range}(f) = B$.

Let f surjects onto B stand for f is surjective onto B. Let a surjective map onto B stand for a map that is surjective onto B.

FOUNDATIONS_08_4195237329108992

Definition 8.2. Let A, B be classes. A surjective map from A to B is a map of A that is surjective onto B.

Let a surjective map from A onto B stand for a surjective map from A to B. Let $f : A \twoheadrightarrow B$ stand for f is a surjective map from A onto B.

FOUNDATIONS_08_1974205941809152

Proposition 8.3. Let B be a class and f be a map to B. f is surjective onto B iff every element of B is a value of f.

Proof. Case f is surjective onto B. Then $B = \mathrm{range}(f)$. Let b be an element of B. Then $b \in \mathrm{range}(f)$. Hence b is a value of f. End.

Case every element of B is a value of f. Let us show that $B \subseteq \mathrm{range}(f)$. Let $b \in B$. Then b is a value of f. Hence $b \in \mathrm{range}(f)$. End.

Let us show that $\mathrm{range}(f) \subseteq B$. Let $b \in \mathrm{range}(f)$. Then b is a value of f. Hence $b \in B$. End. End. □

8.2 Injective maps

FOUNDATIONS_08_605931408719872

Definition 8.4. Let f be a map. f is injective iff for all $a, a' \in \mathrm{dom}(f)$ if $f(a) = f(a')$ then $a = a'$.

Let $f : A \hookrightarrow B$ stand for f is an injective map from A to B.

8.3 Bijective maps

FOUNDATIONS_08_3356670992318464

Definition 8.5. Let A, B be classes. A bijection between A and B is an injective map of A that is surjective onto B.

Let a bijection from A to B stand for a bijection between A and B.

FOUNDATIONS_08_60881194975232

Proposition 8.6. Let A, B be classes and $f : A \hookrightarrow B$. Then f is a bijection between A and $\mathrm{range}(f)$.

Proof. f is injective and surjects onto range(f). Hence f is a bijection between A and range(f). ☐

FOUNDATIONS_08_8188451318923264

Definition 8.7. Let A be a class. A permutation of A is a bijection between A and A.

8.4 Some basic facts

FOUNDATIONS_08_7883784041005056

Proposition 8.8. Let A be a class. Then id_A is a permutation of A.

Proof. (1) id_A is a map on A.

(2) id_A is surjective onto A.
Proof. Let $a \in A$. Then $a = \mathrm{id}_A(a)$. Hence $a \in \mathrm{range}(\mathrm{id}_A)$. Qed.

(3) id_A is injective.
Proof. Let $a, a' \in A$. Assume $\mathrm{id}_A(a) = \mathrm{id}_A(a')$. Then $a = a'$. Qed. ☐

FOUNDATIONS_08_8542698338254848

Proposition 8.9. Let A, B, C be classes and $f : A \twoheadrightarrow B$ and $g : B \twoheadrightarrow C$. Then $g \circ f$ is a surjective map from A onto C.

Proof. $g \circ f$ is a map of A.

Let us show that $g \circ f$ is surjective onto C. Let $c \in C$. Take $b \in B$ such that $c = g(b)$. Take $a \in A$ such that $b = f(a)$. Then $c = g(f(a)) = (g \circ f)(a)$. End. ☐

FOUNDATIONS_08_3367836856614912

Proposition 8.10. Let A, B, C be classes and $f : A \hookrightarrow B$ and $g : B \hookrightarrow C$. Then $g \circ f$ is an injective map from A to C.

Proof. $g \circ f$ is a map of A.

Let us show that $g \circ f$ is injective. Let $a, a' \in A$. Assume $(g \circ f)(a) = (g \circ f)(a')$.

Then $g(f(a)) = g(f(a'))$. Hence $f(a) = f(a')$. Indeed $f(a), f(a') \in B$. Thus $a = a'$.
End. □

FOUNDATIONS_08_6435206693126144

Corollary 8.11. Let A, B, C be classes. Let f be a bijection between A and B and g be a bijection between B and C. Then $g \circ f$ is a bijection between A and C.

FOUNDATIONS_08_2621531811217408

Proposition 8.12. Let A, B be classes and $f : A \hookrightarrow B$ and $X \subseteq A$. Then $f \upharpoonright X$ is injective.

Proof. Let $a, a' \in X$. Assume $(f \upharpoonright X)(a) = (f \upharpoonright X)(a')$. Then $f(a) = f(a')$. Hence $a = a'$. □

FOUNDATIONS_08_647446231252992

Proposition 8.13. Let A, B be classes and $f : A \hookrightarrow B$ and $X \subseteq A$. Then $f \upharpoonright X$ is a bijection between X and $f_*(X)$.

FOUNDATIONS_08_8159443759923200

Corollary 8.14. Let A, B be classes and $f : A \hookrightarrow B$. Then f is a bijection between A and $f_*(A)$.

Chapter 9

Invertible maps and involutions

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9.1 Invertible maps

FOUNDATIONS_09_7776974319648768

Definition 9.1. Let f be a map. An inverse of f is a map g from $\mathrm{range}(f)$ to $\mathrm{dom}(f)$ such that

$$f(a) = b \quad \text{iff} \quad g(b) = a$$

for all $a \in \mathrm{dom}(f)$ and all $b \in \mathrm{dom}(g)$.

FOUNDATIONS_09_3430350086733824

Definition 9.2. Let f be a map. f is invertible iff f has an inverse.

FOUNDATIONS_09_5108611793551360

Lemma 9.3. Let f be a map and g, g' be inverses of f. Then $g = g'$.

Proof. We have $\mathrm{dom}(g) = \mathrm{range}(f) = \mathrm{dom}(g')$.

Let us show that $g(b) = g'(b)$ for all $b \in \mathrm{range}(f)$. Let $b \in \mathrm{range}(f)$. Take $a = g'(b)$. Then $g(b) = a$ iff $f(a) = b$. We have $f(a) = b$ iff $g'(b) = a$. Thus $g(b) = g'(b)$. End. □

FOUNDATIONS_09_6458627204317184

Definition 9.4. Let f be an invertible map. f^{-1} is the inverse of f.

Let f is involutory stand for f is the inverse of f. Let f is selfinverse stand for f is the inverse of f.

9.2 Some basic facts about invertible maps

FOUNDATIONS_09_7840743571849216

Proposition 9.5. Let A, B be classes and $f : A \twoheadrightarrow B$ and $g : B \twoheadrightarrow A$. Then g is the inverse of f iff $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

Proof. Case g is the inverse of f. We have $\mathrm{dom}(g \circ f) = \mathrm{dom}(f) = A = \mathrm{dom}(\mathrm{id}_A)$. For all $a \in A$ we have $(g \circ f)(a) = g(f(a)) = a$. Hence $g \circ f = \mathrm{id}_A$.

We have $\mathrm{dom}(f \circ g) = \mathrm{dom}(g) = B = \mathrm{dom}(\mathrm{id}_B)$. For all $b \in B$ we have $(f \circ g)(b) = f(g(b)) = b$. Hence $f \circ g = \mathrm{id}_B$. End.

Case $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. Then $\mathrm{dom}(g) = B = \mathrm{range}(f)$ and $\mathrm{range}(g) = A = \mathrm{dom}(f)$. Let $a \in \mathrm{dom}(f)$ and $b \in \mathrm{dom}(g)$. If $f(a) = b$ then $g(b) = g(f(a)) = (g \circ f)(a) = \mathrm{id}_A(a) = a$. If $g(b) = a$ then $f(a) = f(g(b)) = (f \circ g)(b) = \mathrm{id}_B(b) = b$. Hence $f(a) = b$ iff $g(b) = a$. End. □

FOUNDATIONS_09_8414736098000896

Proposition 9.6. Let A, B be classes and $f : A \twoheadrightarrow B$. Assume that f is invertible. Then f^{-1} is an invertible surjective map from B onto A such that

$$(f^{-1})^{-1} = f.$$

Proof. f^{-1} is a map from B to A. Indeed $\mathrm{range}(f) = B$ and $\mathrm{dom}(f) = A$. f^{-1} is surjective onto A. Indeed for any $a \in A$ we have $f^{-1}(f(a)) = a$. f^{-1} is the inverse of f. Thus $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$. Therefore f is the inverse of f^{-1}. □

FOUNDATIONS_09_4577560740495360

Proposition 9.7. Let A, B be classes and $f : A \twoheadrightarrow B$. Assume that f is invertible. Then

$$f \circ f^{-1} = \mathrm{id}_B$$

and

$$f^{-1} \circ f = \mathrm{id}_A\,.$$

Proof. f^{-1} is a surjective map from B onto A . f^{-1} is the inverse of f. □

FOUNDATIONS_09_4606651604664320

Proposition 9.8. Let A, B be classes and $f : A \twoheadrightarrow B$ and $a \in A$. Assume that f is invertible. Then

$$f^{-1}(f(a)) = a.$$

Proof. We have $f^{-1}(f(a)) = (f^{-1} \circ f)(a) = \mathrm{id}_A(a) = a$. □

Proposition 9.9. Let A, B be classes and $f : A \twoheadrightarrow B$ and $b \in B$. Assume that f is invertible. Then

$$f(f^{-1}(b)) = b.$$

Proof. We have $f(f^{-1}(b)) = (f \circ f^{-1})(b) = \mathrm{id}_B(b) = b$. □

FOUNDATIONS_09_7619151963095040

Proposition 9.10. Let A, B, C be classes and $f : A \twoheadrightarrow B$ and $g : B \twoheadrightarrow C$. Assume that f and g are invertible. Then $g \circ f$ is invertible and

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

Proof. f^{-1} is a surjective map from B onto A. g^{-1} is a surjective map from C onto B. Take $h = f^{-1} \circ g^{-1}$. Then h is a surjective map from C onto A (by proposition 8.9). $g \circ f$ is a map from A to C.

Let us show that $((g \circ f) \circ h) = \mathrm{id}_C$. We have $f \circ (f^{-1} \circ g^{-1}) = (f \circ f^{-1}) \circ g^{-1}$. Indeed $f \circ (f^{-1} \circ g^{-1})$ and $(f \circ f^{-1}) \circ g^{-1}$ are maps of C. $f \circ h$ is a map from C to B. Hence

$$(g \circ f) \circ h$$

$$= g \circ (f \circ h)$$

$$= g \circ (f \circ (f^{-1} \circ g^{-1}))$$

$$= g \circ ((f \circ f^{-1}) \circ g^{-1})$$

$$= g \circ (\mathrm{id}_B \circ g^{-1})$$

$$= g \circ g^{-1}$$

$$= \mathrm{id}_C\,.$$

End.

Let us show that $h \circ (g \circ f) = \mathrm{id}_A$. We have $(f^{-1} \circ g^{-1}) \circ g = f^{-1} \circ (g^{-1} \circ g)$. $g \circ f$ is a map from A to C. Hence

$$h \circ (g \circ f)$$

$$= (h \circ g) \circ f$$

$$= ((f^{-1} \circ g^{-1}) \circ g) \circ f$$

$$= (f^{-1} \circ (g^{-1} \circ g)) \circ f$$

$$= (f^{-1} \circ \mathrm{id}_B) \circ f$$

$$= f^{-1} \circ f$$

$$= \mathrm{id}_A\,.$$

End.

Thus h is the inverse of $g \circ f$. Indeed $g \circ f$ is a surjective map from A onto C and h is a surjective map from C onto A. □

FOUNDATIONS_09_6374884963778560

Proposition 9.11. Let A, B be classes and $f : A \twoheadrightarrow B$ and $X \subseteq A$. Assume that f is invertible. Then $f \upharpoonright X$ is invertible and

$$(f \upharpoonright X)^{-1} = f^{-1} \upharpoonright (f_*(X)).$$

Proof. $f \upharpoonright X$ is a surjective map from X onto $f_*(X)$. Take $g = f^{-1} \upharpoonright (f_*(X))$. Then g is a map of $f_*(X)$.

Let us show that $X \subseteq \mathrm{range}(g)$. Let $a \in X$. Then $f(a) \in f_*(X)$. Hence $g(f(a)) = f^{-1}(f(a)) = a$. Thus a is a value of g. End.

Let us show that $\mathrm{range}(g) \subseteq X$. Let $a \in \mathrm{range}(g)$. Take $b \in f_*(X)$ such that $a = g(b)$. Take $c \in X$ such that $b = f(c)$. Then $a = (f^{-1} \upharpoonright (f_*(X)))(b) = f^{-1}(b) = f^{-1}(f(c)) = c$. Hence $a \in X$. End.

Hence $\mathrm{range}(g) = X$. Thus g is a surjective map onto X.

Let us show that $g((f \upharpoonright X)(a)) = a$ for all $a \in X$. Let $a \in X$. Then $g((f \upharpoonright X)(a)) = g(f(a)) = (f^{-1} \upharpoonright (f_*(X)))(f(a)) = f^{-1}(f(a)) = a$. End.

Let us show that $((f \restriction X)(g(b))) = b$ for all $b \in f_*(X)$. Let $b \in f_*(X)$. Take $a \in X$ such that $b = f(a)$. We have $g(b) = g(f(a)) = (f^{-1} \restriction (f_*(X)))(f(a)) = f^{-1}(f(a)) = a$. Hence $(f \restriction X)(g(b)) = (f \restriction X)(a) = f(a) = b$. End.

Thus $g \circ (f \restriction X) = \mathrm{id}_X$ and $(f \restriction X) \circ g = \mathrm{id}_{f_*(X)}$. Therefore g is the inverse of $f \restriction X$. □

FOUNDATIONS_09_7726021377785856

Proposition 9.12. Let A, B be classes and $f : A \twoheadrightarrow B$ and $Y \subseteq B$. Assume that f is invertible. Then

$$f^*(Y) = (f^{-1})_*(Y).$$

Proof. We have $(f^{-1})_*(Y) = \{f^{-1}(b) \mid b \in Y\}$ and $f^*(Y) = \{a \in A \mid f(a) \in Y\}$.

Let us show that $f^*(Y) \subseteq (f^{-1})_*(Y)$. Let $a \in f^*(Y)$. Take $b \in Y$ such that $b = f(a)$. Then $f^{-1}(b) = f^{-1}(f(a)) = a$. Hence $a \in (f^{-1})_*(Y)$. End.

Let us show that $f_*^{-1}(Y) \subseteq f^*(Y)$. Let $a \in f_*^{-1}(Y)$. Take $b \in Y$ such that $a = f^{-1}(b)$. Then $f(a) = f(f^{-1}(b)) = b$. Hence $a \in f^*(Y)$. End. □

FOUNDATIONS_09_8607784268464128

Corollary 9.13. Let A, B be classes and $f : A \twoheadrightarrow B$ and $b \in B$. Assume that f is invertible. Then

$$f^*(\{b\}) = \{f^{-1}(b)\}.$$

Proof. $f^*(\{b\}) = f_*^{-1}(\{b\})$. We have $f_*^{-1}(\{b\}) = \{f^{-1}(c) \mid c \in \{b\}\}$. Hence $f_*^{-1}(\{b\}) = \{f^{-1}(b)\}$. □

FOUNDATIONS_09_6777575974109184

Proposition 9.14. Let A, B be classes and $f : A \twoheadrightarrow B$. Then f is invertible iff f is injective.

Proof. Case f is invertible. Let $a, b \in A$. Assume $f(a) = f(b)$. Then $a = f^{-1}(f(a)) = f^{-1}(f(b)) = b$. End.

Case f is injective. Define $g(b) =$ "choose $a \in A$ such that $f(a) = b$ in a" for $b \in B$. Then g is a map from B to A. For all $a \in A$ we have $a = g(f(a))$. Hence g is a surjective map from B onto A. For all $a \in A$ we have $g(f(a)) = a$. For all $b \in B$ we have $f(g(b)) = b$. Hence g is the inverse of f. End. □

FOUNDATIONS_09_5708971514003456

Corollary 9.15. Let A, B be classes and $f : A \twoheadrightarrow B$. Assume that f is invertible. Then f^{-1} is a bijection between B and A.

Proof. f^{-1} is a surjective map from B onto A. f^{-1} is invertible. Hence f^{-1} is injective. Therefore f^{-1} is a bijection between B and A. □

9.3 Involutions

FOUNDATIONS_09_7282039688527872

Definition 9.16. Let A be a class. An involution on A is a selfinverse map f on A.

FOUNDATIONS_09_7944474185433088

Proposition 9.17. Let A be a class. id_A is an involution on A.

Proof. We have $\mathrm{id}_A \circ \mathrm{id}_A = \mathrm{id}_A$. Hence id_A is selfinverse. □

FOUNDATIONS_09_6897019612299264

Proposition 9.18. Let A be a class and f, g be involutions on A. Then $g \circ f$ is an involution on A iff $g \circ f = f \circ g$.

Proof. Case $g \circ f$ is an involution on A. Then $(g \circ f)^{-1} = f^{-1} \circ g^{-1} = f \circ g$. End.

Case $g \circ f = f \circ g$. $f \circ f$, $f \circ g$ and $f \circ g$ are maps on A. Hence

$$(g \circ f) \circ (g \circ f)$$

$$= (g \circ f) \circ (f \circ g)$$

$$= ((g \circ f) \circ f) \circ g$$

$$= (g \circ (f \circ f)) \circ g$$

$$= (g \circ \mathrm{id}_A) \circ g$$

$$= g \circ g$$

$$= \mathrm{id}_A .$$

Thus $g \circ f$ is selfinverse. End. □

FOUNDATIONS_09_5958206868160512

Corollary 9.19. Let A be a class and f be an involutions on A. Then $f \circ f$ is an involution on A.

FOUNDATIONS_09_2314262743613440

Proposition 9.20. Let A be a class and f be an involution on A. Then f is a permutation of A.

Proof. f is an invertible map of A that surjects onto A. Hence f is a bijection between A and A. Thus f is a permutation of A. □

Chapter 10

Sets

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10.1 Sub- and supersets

FOUNDATIONS_10_5530582838673408

Definition 10.1. A proper class is a class that is not a set.

FOUNDATIONS_10_1346889551183872

Definition 10.2. Let A be a class. A subset of A is a subclass of A that is a set.

Let a superset of A stand for a superclass of A that is a set. Let a proper subset of A stand for a proper subclass of A that is a set. Let a proper superset of A stand for a proper superclass of A that is a set.

10.2 Powerclasses

FOUNDATIONS_10_1448589907722240

Definition 10.3. Let A be a class. The powerclass of A is

$$\{x \mid x \text{ is a subset of } A\}.$$

Let $\mathcal{P}(A)$ stand for the powerclass of A.

10.3 Systems of sets

FOUNDATIONS_10_5805323570905088

Definition 10.4. A system of sets is a class X such that every element of X is a set.

FOUNDATIONS_10_1631952387964928

Definition 10.5. A system of nonempty sets is a class X such that every element of X is a nonempty set.

FOUNDATIONS_10_943381479948288

Definition 10.6. Let A be a class. A system of subsets of A is a class X such that every element of X is a subset of A.

FOUNDATIONS_10_8268633648136192

Proposition 10.7. Let A be a class. Then $\emptyset$ is a system of subsets of A.

FOUNDATIONS_10_7546016869908480

Proposition 10.8. Let A be a class. Then $\mathcal{P}(A)$ is a system of subsets of A.

Proposition 10.9. Let X, Y be systems of sets. Then $X \cup Y$ is a system of sets.

Proposition 10.10. Let X, Y be systems of sets. Then $X \cap Y$ is a system of sets.

Proposition 10.11. Let X, Y be systems of sets. Then $X \setminus Y$ is a system of sets.

10.4 Unions

FOUNDATIONS_10_541772562300928

Definition 10.12. Let X be a system of sets. The union over X is

$$\{a \mid a \in x \text{ for some } x \in X\}.$$

Let $\bigcup X$ stand for the union over X.

FOUNDATIONS_10_4872701241982976

Proposition 10.13.

$$\bigcup \emptyset = \emptyset.$$

Proof. $\bigcup \emptyset = \{a \mid a \in x \text{ for some } x \in \emptyset\}$. $\emptyset$ has no elements. Hence there is no object a such that $a \in x$ for some $x \in \emptyset$. Thus $\bigcup \emptyset = \emptyset$. □

FOUNDATIONS_10_2559541585641472

Proposition 10.14. Let x, y be sets. Then

$$\bigcup\{x, y\} = x \cup y.$$

Proof. Let us show that $\bigcup\{x, y\} \subseteq x \cup y$. Let $a \in \bigcup\{x, y\}$. Then a is contained in some element of $\{x, y\}$. Hence $a \in x$ or $a \in y$. Thus $a \in x \cup y$. End.

Let us show that $x \cup y \subseteq \bigcup\{x, y\}$. Let $a \in x \cup y$. Then $a \in x$ or $a \in y$. Hence a is contained in some element of $\{x, y\}$. Therefore $a \in \bigcup\{x, y\}$. End. □

FOUNDATIONS_10_2157223832715264

Corollary 10.15. Let x be a set. Then

$$\bigcup\{x\} = x.$$

10.5 Intersections

FOUNDATIONS_10_2659345095458816

Definition 10.16. Let X be a system of sets. The intersection over X is

$$\{a \mid a \in x \text{ for all } x \in X\}.$$

Let $\bigcap X$ stand for the intersection over X.

FOUNDATIONS_10_2809770322952192

Proposition 10.17. $\bigcap\emptyset$ is the class of all objects.

Proof. Define $V = \{x \mid x \text{ is an object }\}$. We have $\bigcap\emptyset \subseteq V$. Indeed every element of $\bigcap\emptyset$ is an object.

Let us show that $V \subseteq \bigcap\emptyset$. Let $a \in V$. Then a is an object. For every $x \in \emptyset$ we have $a \in x$. Indeed $\emptyset$ has no elements. Thus $a \in \bigcap\emptyset$. End. □

FOUNDATIONS_10_7851827447988224

Proposition 10.18. Let x, y be sets. Then

$$\bigcap\{x, y\} = x \cap y.$$

Proof. Let us show that $\bigcap\{x, y\} \subseteq x \cap y$. Let $a \in \bigcap\{x, y\}$. Then a is contained in every element of $\{x, y\}$. Hence $a \in x$ and $a \in y$. Thus $a \in x \cap y$. End.

Let us show that $x \cap y \subseteq \bigcap\{x, y\}$. Let $a \in x \cap y$. Then $a \in x$ and $a \in y$. Hence a is contained in every element of $\{x, y\}$. Therefore $a \in \bigcap\{x, y\}$. End. □

FOUNDATIONS_10_7239895674257408

Corollary 10.19. Let x be a set. Then

$$\bigcap\{x\} = x.$$

10.6 Classes of functions

FOUNDATIONS_10_5119110467813376

Definition 10.20. Let x, y be sets. $[x \to y]$ is the class of all maps from x to y.

FOUNDATIONS_10_3702893448265728

Proposition 10.21. Let x, y be sets. Then every element of $[x \to y]$ is a function.

10.7 Axioms for mathematics

Definition 10.22. Let A be a class and a be an object and f be a map such that $A \subseteq \mathrm{dom}(f)$. A is inductive regarding a and f iff $a \in A$ and for all $x \in A$ we have $f(x) \in A$.

FOUNDATIONS_10_2362039748001792

Axiom 10.23 (Set existence). There exists a set.

FOUNDATIONS_10_2263707272871936

Axiom 10.24 (Separation). Let A be a class. If there exists a set x such that every element of A is contained in x then A is a set.

FOUNDATIONS_10_7376893816864768

Axiom 10.25 (Pairing). Let a, b be objects. Then $\{a, b\}$ is a set.

FOUNDATIONS_10_5536459412996096

Axiom 10.26 (Union). Let X be a system of sets. If X is a set then $\bigcup X$ is a set.

FOUNDATIONS_10_367388832825344

Axiom 10.27 (Infinity). Let A be a class and $a \in A$ and $f : A \to A$. Then there exists a subset of A that is inductive regarding a and f.

FOUNDATIONS_10_5862230203564032

Axiom 10.28 (Powerset). Let x be a set. Then $\mathcal{P}(x)$ is a set.

Let the powerset of x stand for $\mathcal{P}(x)$.

FOUNDATIONS_10_1897613305577472

Axiom 10.29 (Choice). Let X be a system of nonempty sets. Then there exists a map f such that $\mathrm{dom}(f) = X$ and $f(x) \in x$ for any $x \in X$.

FOUNDATIONS_10_1320008569323520

Axiom 10.30 (Foundation). Let X be a nonempty system of sets. Then X has an element x such that X and x are disjoint.

FOUNDATIONS_10_8142956584239104

Axiom 10.31 (Replacement). Let f be a map and x be a set. Then $f[x]$ is a set.

FOUNDATIONS_10_7781693549182976

Axiom 10.32 (Function). Let f be a map. If $\mathrm{dom}(f)$ is a set then f is a function.

10.8 Consequences of the axioms

FOUNDATIONS_10_5891530432708608

Proposition 10.33. $\emptyset$ is a set.

Proof. Take a set x (by axiom 10.23). Define $A = \{y \in x \mid y \neq y\}$. Then A is a set (by axiom 10.24). We have $A = \emptyset$. Hence $\emptyset$ is a set. □

FOUNDATIONS_10_7556516257202176

Proposition 10.34. Let a be an object. Then $\{a\}$ is a set.

Let the singleton set of a stand for the singleton class of a. Let a singleton set stand for a singleton class.

FOUNDATIONS_10_8408517115379712

Corollary 10.35. Let A be a class that has a unique element. Then A is a set.

FOUNDATIONS_10_4052198354845696

Proposition 10.36. Let x, y be sets. Then $x \cup y$ is a set.

Proof. Take $X = \{x, y\}$. Then X is a set. Hence $\bigcup X$ is a set (by axiom 10.26). Indeed X is a system of sets. We have $x \cup y = \bigcup X$. Thus $x \cup y$ is a set. □

FOUNDATIONS_10_4475839687163904

Proposition 10.37. Let x, y be sets. Then $x \cap y$ is a set.

Proof. We have $x \cap y \subseteq x$. Hence $x \cap y$ is a set (by axiom 10.24). □

FOUNDATIONS_10_7795203882614784

Proposition 10.38. Let x, y be sets. Then $x \setminus y$ is a set.

Proof. We have $x \setminus y \subseteq x$. Hence $x \setminus y$ is a set (by axiom 10.24). □

FOUNDATIONS_10_4458706448154624

Proposition 10.39. Let x, y be sets. Then $x \times y$ is a set.

Proof. $\{a\}$ and $\{a, b\}$ are sets for each $a \in x$ and each $b \in y$. Define $P = \{\{\{a\}, \{a, b\}\} \mid a \in x \text{ and } b \in y\}$.

(1) P is a set.
Proof. Let us show that $P \subseteq \mathcal{P}(\mathcal{P}(x \cup y))$. Let $p \in P$. Consider $a \in x$ and $b \in y$ such that $p = \{\{a\}, \{a, b\}\}$. Then $a, b \in x \cup y$. Hence $\{a\}, \{a, b\} \in \mathcal{P}(x \cup y)$. Thus $\{\{a\}, \{a, b\}\} \in \mathcal{P}(\mathcal{P}(x \cup y))$. End.

$x \cup y$ is a set. Consequently $\mathcal{P}(\mathcal{P}(x \cup y))$ is a set (by axiom 10.28). Therefore P is a set (by axiom 10.24). Qed.

Define $l(p) =$ "choose $a \in x$, choose $b \in y$ such that $p = \{\{a\}, \{a, b\}\}$ in a" for $p \in P$.
Define $r(p) =$ "choose $a \in x$, choose $b \in y$ such that $p = \{\{a\}, \{a, b\}\}$ in b" for $p \in P$.

Define $f(p) = (l(p), r(p))$ for $p \in P$.

Let us show that for any objects u, u', v, v' if $\{\{u\}, \{u, v\}\} = \{\{u'\}, \{u', v'\}\}$ then $u = u'$ and $v = v'$. Let u, u', v, v' be objects. Assume $\{\{u\}, \{u, v\}\} = \{\{u'\}, \{u', v'\}\}$. Then ($\{u\} = \{u'\}$ or $\{u\} = \{u', v'\}$) and ($\{u, v\} = \{u'\}$ or $\{u, v\} = \{u', v'\}$). Thus ($\{u\} = \{u'\}$ and ($\{u, v\} = \{u'\}$ or $\{u, v\} = \{u', v'\}$)) or ($\{u\} = \{u', v'\}$ and ($\{u, v\} = \{u'\}$ or $\{u, v\} = \{u', v'\}$)).

Case $\{u\} = \{u'\}$ and ($\{u, v\} = \{u'\}$ or $\{u, v\} = \{u', v'\}$). We have $\{u\} = \{u'\}$. Hence $u = u'$.

Case $\{u, v\} = \{u'\}$. Then $u = u' = v$. Hence $\{\{u\}, \{u, u\}\} = \{\{u\}, \{u, v'\}\}$ (by 1). Thus $\{\{u\}\} = \{\{u\}, \{u, v'\}\}$. Therefore $\{u\} = \{u, v'\}$. Consequently $v' = u = v$. End.

Case $\{u, v\} = \{u', v'\}$. Then $\{u, v\} = \{u, v'\}$. Hence $v = v'$. End. End.

Case $\{u\} = \{u', v'\}$ and ($\{u, v\} = \{u'\}$ or $\{u, v\} = \{u', v'\}$). We have $\{u\} = \{u', v'\}$. Hence $u = u'$.

Case $\{u, v\} = \{u'\}$. Then $u = v = u'$. Hence $v = v'$. End.

Case $\{u, v\} = \{u', v'\}$. Then $\{u, v\} = \{u, v'\}$. Hence $v = v'$. End. End. End.

Let us show that for any $a \in x$ and any $b \in y$ we have $f(\{\{a\}, \{a, b\}\}) = (a, b)$. Let $a \in x$ and $b \in y$. Take $p = \{\{a\}, \{a, b\}\}$. Then p is a set. Then we can choose $a' \in x$ and $b' \in y$ such that $p = \{\{a'\}, \{a', b'\}\}$ and $l(p) = a'$. Then $a = a'$ and $b = b'$. Hence $l(p) = a$. Choose $a'' \in x$ and $b'' \in y$ such that $p = \{\{a''\}, \{a'', b''\}\}$ and $r(p) = b''$. Then $a = a''$ and $b = b''$. Thus $r(p) = b$. Therefore $f(p) = (a, b)$. End.

(2) $x \times y = f[P]$.
Proof. For all $p \in P$ we have $l(p) \in x$ and $r(p) \in y$. Hence $f(p) \in x \times y$ for all $p \in P$. Therefore $f[P] \subseteq x \times y$.

Let us show that $x \times y \subseteq f[P]$. Let $z \in x \times y$. Take $a \in x$ and $b \in y$ such that $z = (a, b)$. Then $(a, b) = f(\{\{a\}, \{a, b\}\})$. Hence there exists a $p \in P$ such that $(a, b) = f(p)$. Thus $(a, b) \in f[P]$. End.

Consequently $x \times y = f[P]$. Qed.

Thus $x \times y$ is the image of some set under some map. Therefore $x \times y$ is a set (by axiom 10.31). □

FOUNDATIONS_10_5486815207227392

Proposition 10.40. Let X be a nonempty system of sets. Then $\bigcap X$ is a set.

Proof. Take an element x of X. Then $\bigcap X \subseteq x$. Hence $\bigcap X$ is a set (by axiom 10.24). □

FOUNDATIONS_10_7598384349184000

Proposition 10.41. Let f be a map such that $\mathrm{dom}(f)$ is a set. Then $\mathrm{range}(f)$ is a set.

Proof. $\mathrm{range}(f) = f_*(\mathrm{dom}(f))$ and $f_*(\mathrm{dom}(f))$ is a set. Hence $\mathrm{range}(f)$ is a set (by axiom 10.31). □

FOUNDATIONS_10_8631339572002816

Proposition 10.42. Let A be a class and x be a set. Assume that there exists an injective map from A to x. Then A is a set.

Proof. Consider an injective map f from A to x. Then f^{-1} is a bijection between $\mathrm{range}(f)$ and A. $\mathrm{range}(f)$ is a set and A is the image of $\mathrm{range}(f)$ under f^{-1}. Thus A is a set (by axiom 10.31). □

FOUNDATIONS_10_8812282138066944

Proposition 10.43. There exist no sets x, y such that $x \in y$ and $y \in x$.

Proof. Assume the contrary. Take sets x, y such that $x \in y$ and $y \in x$. Consider an element z of $\{x, y\}$ such that $\{x, y\}$ and z are disjoint (by axiom 10.30). Indeed $\{x, y\}$ is a nonempty system of sets. Then we have $z = x$ or $z = y$.

Case $z = x$. Then x and $\{x, y\}$ are disjoint. Hence $y \notin x$. Contradiction. End.

Case $z = y$. Then y and $\{x, y\}$ are disjoint. Hence $x \notin y$. Contradiction. End. □

FOUNDATIONS_10_3086917813927936

Corollary 10.44. Let x be a set. Then $x \notin x$.

FOUNDATIONS_10_4105036244189184

Proposition 10.45. Let x, y be sets. Then $[x \to y]$ is a set.

Proof. Define $R = \{F \in \mathcal{P}(x \times y) \mid$ (for all $a \in x$ there exists a $b \in y$ such that $(a, b) \in F$) and for all $a \in x$ and all $b, b' \in y$ such that $(a, b), (a, b') \in F$ we have $b = b'\}$.

[prover vampire][timelimit 5] Every element of R is a set. Define $h(F) = \lambda a \in x.$ "choose $b \in y$ such that $(a, b) \in F$ in b" for $F \in R$. [prover eprover][/timelimit]

Let us show that $[x \to y] \subseteq \mathrm{range}(h)$. Let $f \in [x \to y]$. Define $F = \{(a, f(a)) \mid a \in x\}$.

Then $F \in R$.
Proof. Define $g(a) = (a, f(a))$ for $a \in x$. Then $F = \mathrm{range}(g)$. Hence F is a set. Thus $F \in \mathcal{P}(x \times y)$. Indeed $F \subseteq x \times y$.

(1) For all $a \in x$ there exists a $b \in y$ such that $(a, b) \in F$.

(2) For all $a \in x$ and all $b, b' \in y$ such that $(a, b), (a, b') \in F$ we have $b = b'$. End.

We have $\mathrm{dom}(f) = x = \mathrm{dom}(h(F))$. For each $a \in x$ we have $h(F)(a) = f(a)$. Hence $f = h(F)$. Thus $f \in \mathrm{range}(h)$. End.

Therefore $[x \to y]$ is a set. Indeed R is a set. □

Chapter 11

Binary relations

File: `foundations/sections/11_binary-relations.ftl.tex`

[readtex `foundations/sections/10_sets.ftl.tex`]

FOUNDATIONS_11_6429308924985344

Definition 11.1. A binary relation is a class R such that every element of R is a pair.

11.1 Properties of relations

Reflexivity

FOUNDATIONS_11_1126092393938944

Definition 11.2. Let R be a binary relation and A be a class. R is reflexive on A iff for all $a \in A$ we have $(a, a) \in R$.

Irreflexivity

FOUNDATIONS_11_365656446861312

Definition 11.3. Let R be a binary relation and A be a class. R is irreflexive on A iff for no $a \in A$ we have $(a, a) \in R$.

Symmetry

FOUNDATIONS_11_2056300137545728

Definition 11.4. Let R be a binary relation and A be a class. R is symmetric on A iff for all $a, b \in A$ if $(a, b) \in R$ then $(b, a) \in R$.

Antisymmetry

FOUNDATIONS_11_8301693043212288

Definition 11.5. Let R be a binary relation and A be a class. R is antisymmetric on A iff for all distinct $a, b \in A$ we have $(a, b) \notin R$ or $(b, a) \notin R$.

Asymmetry

FOUNDATIONS_11_6895428727472128

Definition 11.6. Let R be a binary relation and A be a class. R is asymmetric on A iff for all $a, b \in A$ if $(a, b) \in R$ then $(b, a) \notin R$.

Transitivity

FOUNDATIONS_11_5377309666181120

Definition 11.7. Let R be a binary relation and A be a class. R is transitive on A iff for all $a, b, c \in A$ if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

Connectedness

FOUNDATIONS_11_5902056743239680

Definition 11.8. Let R be a binary relation and A be a class. R is connected on A iff for all distinct $a, b \in A$ we have $(a, b) \in R$ or $(b, a) \in R$.

Strong connectedness

FOUNDATIONS_11_6492592562765824

Definition 11.9. Let R be a binary relation and A be a class. R is strongly connected on A iff for all $a, b \in A$ we have $(a, b) \in R$ or $(b, a) \in R$.

11.2 Order relations

Preorders.

FOUNDATIONS_11_4005024520732672

Definition 11.10. Let A be a class. A preorder on A is a binary relation that is reflexive on A and transitive on A.

Partial orders.

FOUNDATIONS_11_2162776243961856

Definition 11.11. Let A be a class. A partial order on A is a binary relation R that is reflexive on A and antisymmetric on A and transitive on A.

Let A is partially ordered by R stand for R is a partial order on A.

Strict partial orders.

FOUNDATIONS_11_4067384857985024

Definition 11.12. Let A be a class. A strict preorder on A is a binary relation that is irreflexive on A and transitive on A.

Let A is strictly preordered by R stand for R is a strict preorder on A.

FOUNDATIONS_11_5567849812721664

Proposition 11.13. Let A be a class. Any strict preorder on A is antisymmetric on A.

Let a strict partial order on A stand for a strict preorder on A. Let A is strictly partially ordered by R stand for R is a strict partial order on A.

Total orders.

FOUNDATIONS_11_5872706501214208

Definition 11.14. Let A be a class. A total order on A is a partial order on A that is connected on A.

Let A is totally ordered by R stand for R is a total order on A.

Let a linear order on A stand for a total order on A. Let A is linearly ordered by R stand for R is a linear order on A.

Strict total orders.

FOUNDATIONS_11_5840248768561152

Definition 11.15. Let A be a class. A strict total order on A is a strict partial order on A that is connected on A.

Let A is stritcly totally ordered by R stand for R is a strict total order on A.

Let a strict linear order on A stand for a strict total order on A. Let A is strictly linearly ordered by R stand for R is a strict linear order on A.

11.3 Well-founded relations

FOUNDATIONS_11_2729326472593408

Definition 11.16. Let A be a class and R be a binary relation. A least element of A regarding R is an element a of A such that there exists no $x \in A$ such that $(x, a) \in R$.

FOUNDATIONS_11_2420057567133696

Definition 11.17. Let A be a class and R be a binary relation. R is wellfounded on A iff every nonempty subclass of A has a least element regarding R.

FOUNDATIONS_11_3262141912055808

Definition 11.18. Let A be a class and R be a binary relation. R is strongly wellfounded on A iff R is wellfounded on A and for all $b \in A$ there exists a set X such that

$$X = \{a \in A \mid (a, b) \in R\}.$$

FOUNDATIONS_11_6149137814781952

Definition 11.19. Let A be a class. A wellorder on A is a strict linear order on A that is wellfounded on A.

FOUNDATIONS_11_8163723743068160

Definition 11.20. Let A be a class. A strong wellorder on A is a strict linear order on A that is strongly wellfounded on A.

11.4 Epsilon induction

FOUNDATIONS_11_4800525813940224

Definition 11.21.

$$\in = \{(a, x) \mid x \text{ is a set that contains } a\}.$$

FOUNDATIONS_11_5668859243659264

Proposition 11.22. $\in$ is strongly wellfounded on any system of sets.

Proof. Let X be a system of sets.

(1) $\in$ is wellfounded on X.
Proof. Let A be a nonempty subclass of X. Take an element x of A such that A and x are disjoint. Then x is a least element of A regarding $\in$. Indeed for any $a \in A$ if $a \in x$ then $a \in A \cap x$. Qed.

(2) For all $x \in X$ there exists a set Y such that $Y = \{y \in X \mid (y, x) \in \in\}$.
Proof. Let $x \in X$. Define $Y = \{y \in X \mid (y, x) \in \in\}$. Then $Y = \{y \in X \mid y \in x\}$. Hence Y is a subclass of x. Thus Y is a set. Qed. □

FOUNDATIONS_11_6337807438053376

Corollary 11.23. Every nonempty system of sets has a least element regarding $\in$.

FOUNDATIONS_11_2812087589928960

Proposition 11.24. Let Φ be a class. (Induction hypothesis) Assume that for all sets x if Φ contains every element of x that is a set then Φ contains x. Then Φ contains every set.

Proof. Assume the contrary. Define $M = \{x \mid x \text{ is a set such that } x \notin \Phi\}$. Then M is nonempty. Hence we can take a least element x of M regarding $\in$. Then x is a set such that every element of x that is a set is contained in Φ. Thus Φ contains x (by induction hypothesis). Contradiction. □

Chapter 12

Fixed points

File: `foundations/sections/12_fixed-points.ftl.tex`

[readtex `foundations/sections/10_sets.ftl.tex`]

FOUNDATIONS_12_2177076576649216

Definition 12.1. Let f be a map. A fixed point of f is an element x of $\mathrm{dom}(f)$ such that $f(x) = x$.

FOUNDATIONS_12_1394550966845440

Definition 12.2. A map between systems of sets is a map from some system of sets to some system of sets.

FOUNDATIONS_12_3290499861446656

Definition 12.3. Let f be a map between systems of sets. f is subset preserving iff for all $x, y \in \mathrm{dom}(f)$

$$x \subseteq y \quad \text{implies} \quad f(x) \subseteq f(y).$$

FOUNDATIONS_12_8420450166112256

Theorem 12.4 (Knaster-Tarski). Let x be a set. Let f be a subset preserving map from $\mathcal{P}(x)$ to $\mathcal{P}(x)$. Then f has a fixed point.

Proof. (1) Define $A = \{y \mid y \subseteq x \text{ and } y \subseteq f(y)\}$. Then A is a subset of $\mathcal{P}(x)$. We have $\bigcup A \in \mathcal{P}(x)$.

Let us show that (2) $\bigcup A \subseteq f(\bigcup A)$. Let $u \in \bigcup A$. Take $y \in A$ such that $u \in y$. Then $u \in f(y)$. We have $y \subseteq \bigcup A$. Hence $f(y) \subseteq f(\bigcup A)$. Thus $f(y) \subseteq f(\bigcup A)$. Therefore $u \in f(\bigcup A)$. End.

Then $f(\bigcup A) \in A$ (by 1). Indeed $f(\bigcup A) \subseteq x$. (3) Hence $f(\bigcup A) \subseteq \bigcup A$. Indeed every element of $f(\bigcup A)$ is an element of some element of A.

Thus $f(\bigcup A) = \bigcup A$ (by 2, 3). □

Chapter 13

Equinumerosity

File: `foundations/sections/13_equinumerosity.ftl.tex`

[readtex `foundations/sections/12_fixed-points.ftl.tex`]

FOUNDATIONS_13_4578620297183232

Definition 13.1. Let A, B be classes. A is equinumerous to B iff there exists a bijection between A and B.

FOUNDATIONS_13_3703161885818880

Proposition 13.2. Let A be a class. Then A is equinumerous to A.

Proof. id_A is a bijection between A and A. □

FOUNDATIONS_13_8050301789536256

Proposition 13.3. Let A, B be classes. If A and B are equinumerous then B and A are equinumerous.

Proof. Assume that A and B are equinumerous. Take a bijection f between A and B. Then f^{-1} is a bijection between B and A. Hence B and A are equinumerous. □

FOUNDATIONS_13_3609912414306304

Proposition 13.4. Let A, B, C be classes. If A and B are equinumerous and B and C are equinumerous then A and C are equinumerous.

Proof. Assume that A and B are equinumerous and B and C are equinumerous. Take a bijection f between A and B and a bijection g between B and C. Then $g \circ f$ is a bijection between A and C. Hence A and C are equinumerous. □

FOUNDATIONS_13_1913663275401216

Theorem 13.5 (Cantor-Schröder-Bernstein). Let x, y be sets. Then x and y are equinumerous iff there exists an injective map from x to y and there exists an injective map from y to x.

Proof. Case x and y are equinumerous. Take a bijection f between x and y. Then f^{-1} is a bijection between y and x. Hence f is an injective map from x to y and f^{-1} is an injective map from y to x. End.

Case there exists an injective map from x to y and there exists an injective map from y to x. Take an injective map f from x to y. Take an injective map g from y to x. We have $y \setminus f[a] \subseteq y$ for any $a \in \mathcal{P}(x)$.

(1) Define $h(a) = x \setminus g[y \setminus f[a]]$ for $a \in \mathcal{P}(x)$.

h is a map from $\mathcal{P}(x)$ to $\mathcal{P}(x)$. Indeed $h(a)$ is a subset of x for each subset a of x.

Let us show that h is subset preserving. Let u, v be subsets of x. Assume $u \subseteq v$. Then $f[u] \subseteq f[v]$. Hence $y \setminus f[v] \subseteq y \setminus f[u]$. Thus $g[y \setminus f[v]] \subseteq g[y \setminus f[u]]$. Indeed $y \setminus f[v]$ and $y \setminus f[u]$ are subsets of y. Therefore $x \setminus g[y \setminus f[u]] \subseteq x \setminus g[y \setminus f[v]]$. Consequently $h[u] \subseteq h[v]$. End.

Hence we can take a fixed point c of h (by theorem 12.4).

(2) Define $F(u) = f(u)$ for $u \in c$.

We have $c = h(c)$ iff $x \setminus c = g[y \setminus f[c]]$. g^{-1} is a bijection between range(g) and y. Thus $x \setminus c = g[y \setminus f[c]] \subseteq$ range(g). Therefore $x \setminus c$ is a subset of dom(g^{-1}).

(3) Define $G(u) = g^{-1}(u)$ for $u \in x \setminus c$.

F is a bijection between c and range(F). G is a bijection between $x \setminus c$ and range(G).

Define

$$H(u) = \begin{cases} F(u) & : u \in c \\ G(u) & : u \notin c \end{cases}$$

for $u \in x$.

Let us show that H is a map to y. dom(H) is a set and every value of H is an object.

Hence H is a map.

Let us show that every value of H lies in y. Let v be a value of H. Take $u \in x$ such that $H(u) = v$. If $u \in c$ then $v = H(u) = F(u) = f(u) \in y$. If $u \notin c$ then $v = H(u) = G(u) = g^{-1}(u) \in y$. End. End.

(4) H is surjective onto y. Indeed we can show that every element of y is a value of H. Let $v \in y$.

Case $v \in f[c]$. Take $u \in c$ such that $f(u) = v$. Then $F(u) = v$. End.

Case $v \notin f[c]$. Then $v \in y \setminus f[c]$. Hence $g(v) \in g[y \setminus f[c]]$. Thus $g(v) \in x \setminus h(c)$. We have $g(v) \in x \setminus c$. Therefore we can take $u \in x \setminus c$ such that $G(u) = v$. Then $v = H(u)$. End. End.

(5) H is injective. Indeed we can show that for all $u, v \in x$ if $u \neq v$ then $H(u) \neq H(v)$. Let $u, v \in x$. Assume $u \neq v$.

Case $u, v \in c$. Then $H(u) = F(u)$ and $H(v) = F(v)$. We have $F(u) \neq F(v)$. Hence $H(u) \neq H(v)$. End.

Case $u, v \notin c$. Then $H(u) = G(u)$ and $H(v) = G(v)$. We have $G(u) \neq G(v)$. Hence $H(u) \neq H(v)$. End.

Case $u \in c$ and $v \notin c$. Then $H(u) = F(u)$ and $H(v) = G(v)$. Hence $v \in g[y \setminus f[c]]$. We have $G(v) \in y \setminus F[c]$. Thus $G(v) \neq F(u)$. End.

Case $u \notin c$ and $v \in c$. Then $H(u) = G(u)$ and $H(v) = F(v)$. Hence $u \in g[y \setminus f[c]]$. We have $G(u) \in y \setminus f[c]$. Thus $G(u) \neq F(v)$. End. End.

Consequently H is a bijection between x and y (by 4, 5). Therefore x and y are equinumerous. End. $\square$