The Knaster-Tarski fixed point theorem and the Cantor-Schröder-Bernstein Theorem

ℕaproche formalization:

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This is a formalization of the *Knaster-Tarski Fixed Point Theorem* (i.e. of the assertion that every subset-preserving map has a fixed point) and of the *Cantor-Schröder-Bernstein Theorem* (i.e. of the fact that two sets are equinumerous iff they can be embedded into each other), where the proof of the latter theorem is based on the former one, as in [1, p. 530].

On mid-range hardware ℕaproche needs approximately 3 Minutes to verify this formalization plus approximately 7 minutes to verify the library files it depends on.

[readtex `foundations/sections/10_sets.ftl.tex`]

The Knaster-Tarski fixed point theorem

Definition. Let h be a map. A fixed point of h is an element u of $\mathrm{dom}(h)$ such that $h(u) = u$.

Definition. A map between systems of sets is a map from some system of sets to some system of sets.

Definition. Let h be a map between systems of sets. h preserves subsets iff for all $u, v \in \mathrm{dom}(h)$ we have

$$u \subseteq v \implies h(u) \subseteq h(v).$$

Theorem (Knaster-Tarski). Let x be a set. Let h be a map from $\mathcal{P}(x)$ to $\mathcal{P}(x)$ that preserves subsets. Then h has a fixed point.

Proof. (1) Define $A = \{y \mid y \subseteq x \text{ and } y \subseteq h(y)\}$. Then A is a subset of $\mathcal{P}(x)$. We have $\bigcup A \in \mathcal{P}(x)$.

Let us show that (2) $\bigcup A \subseteq h(\bigcup A)$. Let $u \in \bigcup A$. Take $y \in A$ such that $u \in y$. Then $u \in h(y)$. We have $y \subseteq \bigcup A$. Hence $h(y) \subseteq h(\bigcup A)$. Thus

$h(y) \subseteq h(\bigcup A)$. Therefore $u \in h(\bigcup A)$. End.

Then $h(\bigcup A) \in A$ (by 1). Indeed $h(\bigcup A) \subseteq x$. (3) Hence $h(\bigcup A) \subseteq \bigcup A$. Indeed every element of $h(\bigcup A)$ is an element of some element of A.

Thus $h(\bigcup A) = \bigcup A$ (by 2, 3). □

The Cantor-Schröder-Bernstein theorem

Definition. Let x, y be sets. x and y are equinumerous iff there exists a bijection between x and y.

Theorem (Cantor-Schröder-Bernstein). Let x, y be sets. x and y are equinumerous iff there exists a injective map from x to y and there exists an injective map from y to x.

Proof. Case x and y are equinumerous. Take a bijection f between x and y. Then f^{-1} is a bijection between y and x. Hence f is an injective map from x to y and f^{-1} is an injective map from y to x. End.

Case there exists an injective map from x to y and there exists an injective map from y to x. Take an injective map f from x to y. Take an injective map g from y to x. We have $y \setminus f[a] \subseteq y$ for any $a \in \mathcal{P}(x)$.

(1) Define $h(a) = x \setminus g[y \setminus f[a]]$ for $a \in \mathcal{P}(x)$.

h is a map from $\mathcal{P}(x)$ to $\mathcal{P}(x)$. Indeed $h(a)$ is a subset of x for each subset a of x.

Let us show that h preserves subsets. Let u, v be subsets of x. Assume $u \subseteq v$. Then $f[u] \subseteq f[v]$. Hence $y \setminus f[v] \subseteq y \setminus f[u]$. Thus $g[y \setminus f[v]] \subseteq g[y \setminus f[u]]$. Indeed $y \setminus f[v]$ and $y \setminus f[u]$ are subsets of y. Therefore $x \setminus g[y \setminus f[u]] \subseteq x \setminus g[y \setminus f[v]]$. Consequently $h[u] \subseteq h[v]$. End.

Hence we can take a fixed point c of h (by Knaster-Tarski).

(2) Define $F(u) = f(u)$ for $u \in c$.

We have $c = h(c)$ iff $x \setminus c = g[y \setminus f[c]]$. g^{-1} is a bijection between range(g) and y. Thus $x \setminus c = g[y \setminus f[c]] \subseteq \text{range}(g)$. Therefore $x \setminus c$ is a subset of dom(g^{-1}).

(3) Define $G(u) = g^{-1}(u)$ for $u \in x \setminus c$.

F is a bijection between c and range(F). G is a bijection between $x \setminus c$ and range(G).

Define

$$H(u) = \begin{cases} F(u) & : u \in c \\ G(u) & : u \notin c \end{cases}$$

for $u \in x$.

Let us show that H is a map to y. $\mathrm{dom}(H)$ is a set and every value of H is an object. Hence H is a map.

Let us show that every value of H lies in y. Let v be a value of H. Take $u \in x$ such that $H(u) = v$. If $u \in c$ then $v = H(u) = F(u) = f(u) \in y$. If $u \notin c$ then $v = H(u) = G(u) = g^{-1}(u) \in y$. End. End.

(4) H is surjective onto y. Indeed we can show that every element of y is a value of H. Let $v \in y$.

Case $v \in f[c]$. Take $u \in c$ such that $f(u) = v$. Then $F(u) = v$. End.

Case $v \notin f[c]$. Then $v \in y \setminus f[c]$. Hence $g(v) \in g[y \setminus f[c]]$. Thus $g(v) \in x \setminus h(c)$. We have $g(v) \in x \setminus c$. Therefore we can take $u \in x \setminus c$ such that $G(u) = v$. Then $v = H(u)$. End. End.

(5) H is injective. Indeed we can show that for all $u, v \in x$ if $u \neq v$ then $H(u) \neq H(v)$. Let $u, v \in x$. Assume $u \neq v$.

Case $u, v \in c$. Then $H(u) = F(u)$ and $H(v) = F(v)$. We have $F(u) \neq F(v)$. Hence $H(u) \neq H(v)$. End.

Case $u, v \notin c$. Then $H(u) = G(u)$ and $H(v) = G(v)$. We have $G(u) \neq G(v)$. Hence $H(u) \neq H(v)$. End.

Case $u \in c$ and $v \notin c$. Then $H(u) = F(u)$ and $H(v) = G(v)$. Hence $v \in g[y \setminus f[c]]$. We have $G(v) \in y \setminus F[c]$. Thus $G(v) \neq F(u)$. End.

Case $u \notin c$ and $v \in c$. Then $H(u) = G(u)$ and $H(v) = F(v)$. Hence $u \in g[y \setminus f[c]]$. We have $G(u) \in y \setminus f[c]$. Thus $G(u) \neq F(v)$. End. End.

Consequently H is a bijection between x and y (by 4, 5). Therefore x and y are equinumerous. End. $\square$

References

[1] Bernd S. W. Schröder, *The fixed point property for ordered sets*; Springer, *Arabian Journal of Mathematics*, vol. 1, p. 529–547