Chapter 1

Cardinality

File: `arithmetic/sections/11_cardinality.ftl.tex`

[readtex `foundations/sections/13_equinumerosity.ftl.tex`]

[readtex `arithmetic/sections/04_ordering.ftl.tex`]

1.1 Subsections of the natural numbers

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Definition 1.1. Let n, m be natural numbers. $\{n, \dots, m\} = \{k \in \mathbb{N} \mid n \leq k \leq m\}$.

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Proposition 1.2. Let n, m be natural numbers. If $\{1, \dots, n\} = \{1, \dots, m\}$ then $n = m$.

Proof. Assume $\{1, \dots, n\} = \{1, \dots, m\}$.

Case $n = 0$. Then $\{1, \dots, n\} = \emptyset$. Thus $\{1, \dots, m\} = \emptyset$. Hence there exists no $k \in \mathbb{N}$ such that $1 \leq k \leq m$. Therefore $m = 0$. Consequently $n = m$. End.

Case $m = 0$. Then $\{1, \dots, m\} = \emptyset$. Thus $\{1, \dots, n\} = \emptyset$. Hence there exists no $k \in \mathbb{N}$ such that $1 \leq k \leq n$. Therefore $n = 0$. Consequently $n = m$. End.

Case $n, m \geq 1$. For all $k \in \mathbb{N}$ we have $1 \leq k \leq n$ iff $1 \leq k \leq m$. Hence for all $k \in \mathbb{N}$ we have $k \leq n$ iff $k \leq m$.

Let us show by contradiction that $n = m$. Suppose $n \neq m$. Then $n > m$ or $m > n$.

Case $n > m$. Take $k = m + 1$. Then $k \leq n$ and $k \nleq m$. Hence it is wrong that $k \leq n$ iff $k \leq m$. Contradiction. End.

Case $m > n$. Take $k = n + 1$. Then $k \leq m$ and $k \nleq m$. Hence it is wrong that $k \leq n$ iff $k \leq m$. Contradiction. End. End. End. □

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Proposition 1.3. Let n be a natural number. Then $\{1, \ldots, n+1\} = \{1, \ldots, n\} \cup \{n + 1\}$.

Proof. We have $\{1, \ldots, n + 1\} \subseteq \{1, \ldots, n\} \cup \{n + 1\}$ and $\{1, \ldots, n\} \cup \{n + 1\} \subseteq \{1, \ldots, n + 1\}$. □

1.2 Finite and infinite sets

Finite sets

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Definition 1.4. Let X be a set. X is finite iff there exists a natural number n such that X is equinumerous to $\{1, \ldots, n\}$.

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Proposition 1.5. Let X, Y be sets. If X is finite and Y is equinumerous to X then Y is finite.

Proof. Assume that X is finite and Y is equinumerous to X. Take a natural number n and a bijection f between $\{1, \ldots, n\}$ and X and a bijection g between X and Y. Then $g \circ f$ is a bijection between $\{1, \ldots, n\}$ and Y (by **??**). Indeed X, Y are classes. Hence Y is finite. □

Infinite sets

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Definition 1.6. Let X be a set. X is infinite iff X is not finite.

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Proposition 1.7. Let X, Y be sets. If X is infinite and Y is equinumerous to X then Y is infinite.

Proof. Assume that Y is equinumerous to X. If Y is finite then X is finite. Hence if X is infinite then Y is infinite. □

The cardinality of a set

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Signature 1.8. ∞ is an object that is not a natural number.

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Definition 1.9. Let X be a set. The cardinality of X is the object κ such that

(if X is finite then κ is the natural number n such that X is equinumerous to $\{1, \ldots, n\}$) and

if X is infinite then $\kappa = \infty$.

Let $|X|$ stand for the cardinality of X.

Let X has finitely many elements stand for $|X| \in \mathbb{N}$. Let X has infinitely many elements stand for $|X| = \infty$.

Let X has exactly n elements stand for $|X| = n$. Let X has at most n elements stand for $|X| \leq n$. Let X has at least n elements stand for $|X| \geq n$.

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Proposition 1.10. Let X be a set. X is empty iff $|X| = 0$.

Proof. Case X is empty. Then $X = \emptyset = \{1, \ldots, 0\}$. Hence X is equinumerous to $\{1, \ldots, 0\}$. Thus $|X| = 0$. End.

Case $|X| = 0$. Then X is equinumerous to $\{1, \ldots, 0\}$. $\{1, \ldots, 0\} = \emptyset$. Thus $X = \emptyset$.

End. □

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Proposition 1.11. Let X be a set. X is a singleton set iff $|X| = 1$.

Proof. Case X is a singleton set. Consider an object a such that $X = \{a\}$. Define $f(x) = 1$ for $x \in X$. Then f is a bijection between X and $\{1\}$. We have $\{1\} = \{1, \ldots, 1\}$. Hence $|X| = 1$. End.

Case $|X| = 1$. Take a bijection f between $\{1, \ldots, 1\}$ and X. We have $\{1, \ldots, 1\} = \{1\}$. Hence $X = \{f(1)\}$. End. □

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Proposition 1.12. Let X be a set. X is an unordered pair iff $|X| = 2$.

Proof. Case X is an unordered pair. Consider distinct objects a, b such that $X = \{a, b\}$. Define

$$f(x) = \begin{cases} 1 & : x = a \\ 2 & : x = b \end{cases}$$

for $x \in X$. Then f is a bijection between X and $\{1, 2\}$. We have $\{1, \ldots, 2\} = \{1, 2\}$. Hence $|X| = 2$. End.

Case $|X| = 2$. Take a bijection f between $\{1, \ldots, 2\}$ and X. We have $\{1, \ldots, 2\} = \{1, 2\}$. Hence $X = \{f(1), f(2)\}$. End. □

1.3 Countable and uncountable sets

Countably infinite sets

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Definition 1.13. Let X be a set. X is countably infinite iff X is equinumerous to $\mathbb{N}$.

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Proposition 1.14. Let X, Y be sets. If X is countably infinite and Y is equinumerous to X then Y is countably infinite.

Proof. Assume that X is countably infinite and Y is equinumerous to X. Take a bijection f between $\mathbb{N}$ and X and a bijection g between X and Y. Then $g \circ f$ is a bijection between $\mathbb{N}$ and Y (by **??**). Indeed X, Y are classes. Hence Y is countably infinite. □

Countable sets

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Definition 1.15. Let X be a set. X is countable iff X is finite or X is countably infinite.

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Proposition 1.16. Let X, Y be sets. If X is countable and Y is equinumerous to X then Y is countable.

Proof. Assume that X is countable and Y is equinumerous to X. If X is finite then Y is finite. If X is countably infinite then Y is countably infinite. Hence Y is countable. □

Uncountable sets

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Definition 1.17. Let X be a set. X is uncountable iff X is not countable.

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Proposition 1.18. Let X, Y be sets. If X is uncountable and Y is equinumerous to X then Y is uncountable.

Proof. Assume that Y is equinumerous to X. If Y is countable then X is countable. Hence if X is uncountable then Y is uncountable. □

1.4 Systems of sets

Definitions

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Definition 1.19. A system of finite sets is a system of sets X such that every element of X is finite.

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Definition 1.20. A system of countably infinite sets is a system of sets X such that every element of X is countably infinite.

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Definition 1.21. A system of countable sets is a system of sets X such that every element of X is countable.

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Definition 1.22. A system of uncountable sets is a system of sets X such that every element of X is uncountable.

Closure under unions

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Definition 1.23. Let X be a system of sets. X is closed under arbitrary unions iff $\bigcup U \in X$ for every nonempty subset U of X.

Let X is closed under unions stand for X is closed under arbitrary unions.

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Definition 1.24. Let X be a system of sets. X is closed under countable unions iff $\bigcup U \in X$ for every nonempty countable subset U of X.

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Definition 1.25. Let X be a system of sets. X is closed under finite unions iff $\bigcup U \in X$ for every nonempty finite subset U of X.

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Proposition 1.26. Let X be a system of sets. X is closed under finite unions iff $U \cup V \in X$ for every $U, V \in X$.

Proof. Case X is closed under finite unions. Let $U, V \in X$. Then $\{U, V\}$ is a nonempty finite subset of X. Hence $U \cup V = \bigcup\{U, V\} \in X$. End.

Case $U \cup V \in X$ for every $U, V \in X$. Define $\Phi = \{n \in \mathbb{N} \mid \bigcup U \in X$ for every nonempty subset U of X such that $|U| = n\}$.

(1) Φ contains 0.

(2) For every $n \in \Phi$ we have $n + 1 \in \Phi$.
Proof. Let $n \in \Phi$. Then $\bigcup U \in X$ for every nonempty subset U of X such that $|U| = n$.

Let us show that $\bigcup U \in X$ for every nonempty subset U of X such that $|U| = n + 1$.

Case $n = 0$. Obvious.

Case $n \neq 0$. Let U be a nonempty subset of X such that $|U| = n+1$. Take a bijection f between $\{1, \dots, n+1\}$ and U. We have $\{1, \dots, n+1\} = \{1, \dots, n\} \cup \{n+1\}$. Take $V = f[\{1, \dots, n\}]$. We have $\{1, \dots, n\} \subseteq \{1, \dots, n + 1\}$.

Let us show that $V \subseteq U$. Let $x \in V$. Take $k \in \{1, \dots, n\}$ such that $x = f(k)$. Hence $x \in U$. End.

V is a nonempty set. Hence V is a nonempty subset of X. U is a class and $f : \{1, \dots, n + 1\} \hookrightarrow U$. [prover vampire] Hence $f \restriction \{1, \dots, n\}$ is a bijection between $\{1, \dots, n\}$ and V (by **??**). [prover eprover] Thus $|V| = n$. Consequently $\bigcup V \in X$. We have $U = V \cup \{f(n+1)\}$. Indeed $U = f[\{1, \dots, n+1\}] = f[\{1, \dots, n\} \cup \{n+1\}] = f[\{1, \dots, n\}] \cup f[\{n + 1\}] = f[\{1, \dots, n\}] \cup \{f(n + 1)\}$.

Let us show that $\bigcup(A \cup B) = (\bigcup A) \cup (\bigcup B)$ for any nonempty systems of sets A, B. Let A, B be nonempty systems of sets. $\bigcup(A \cup B) \subseteq (\bigcup A) \cup (\bigcup B)$. $((\bigcup A) \cup (\bigcup B)) \subseteq \bigcup(A \cup B)$. End.

Hence $\bigcup U = \bigcup(V \cup \{f(n + 1)\}) = (\bigcup V) \cup (\bigcup\{f(n + 1)\}) = (\bigcup V) \cup f(n + 1) \in X$. Indeed V and $\{f(n + 1)\}$ are nonempty systems of sets. End. End. Qed.

Therefore Φ contains every natural number. Thus $\bigcup U \in X$ for every nonempty finite subset U of X. Consequently X is closed under finite unions. End. ☐

Closure under intersections

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Definition 1.27. Let X be a system of sets. X is closed under arbitrary intersections iff $\bigcap U \in X$ for every nonempty subset U of X.

Let X is closed under intersections stand for X is closed under arbitrary intersections.

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Definition 1.28. Let X be a system of sets. X is closed under countable intersections iff $\bigcap U \in X$ for every nonempty countable subset U of X.

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Definition 1.29. Let X be a system of sets. X is closed under finite intersections iff $\bigcap U \in X$ for every nonempty finite subset U of X.

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Proposition 1.30. Let X be a system of sets. X is closed under finite intersections iff $U \cap V \in X$ for every $U, V \in X$.

Proof. Case X is closed under finite intersections. Let $U, V \in X$. Then $\{U, V\}$ is a nonempty finite subset of X. Hence $U \cap V = \bigcap\{U, V\} \in X$. End.

Case $U \cap V \in X$ for every $U, V \in X$. Define $\Phi = \{n \in \mathbb{N} \mid \bigcap U \in X$ for every nonempty subset U of X such that $|U| = n\}$.

(1) Φ contains 0.

(2) For every $n \in \Phi$ we have $n + 1 \in \Phi$.
Proof. Let $n \in \Phi$. Then $\bigcap U \in X$ for every nonempty subset U of X such that $|U| = n$.

Let us show that $\bigcap U \in X$ for every nonempty subset U of X such that $|U| = n + 1$.

Case $n = 0$. Obvious.

Case $n \neq 0$. Let U be a nonempty subset of X such that $|U| = n+1$. Take a bijection f between $\{1, \ldots, n+1\}$ and U. We have $\{1, \ldots, n+1\} = \{1, \ldots, n\} \cup \{n+1\}$. Take $V = f[\{1, \ldots, n\}]$. We have $\{1, \ldots, n\} \subseteq \{1, \ldots, n + 1\}$.

Let us show that $V \subseteq U$. Let $x \in V$. Take $k \in \{1, \ldots, n\}$ such that $x = f(k)$. Hence $x \in U$. End.

V is a nonempty set. Hence V is a nonempty subset of X. U is a class and $f : \{1, \ldots, n+1\} \hookrightarrow U$. [prover vampire] Hence $f \restriction \{1, \ldots, n\}$ is a bijection between $\{1, \ldots, n\}$ and V (by **??**). [prover eprover] Thus $|V| = n$. Consequently $\bigcap V \in X$. We have $U = V \cup \{f(n+1)\}$. Indeed $U = f[\{1, \ldots, n+1\}] = f[\{1, \ldots, n\} \cup \{n+1\}] = f[\{1, \ldots, n\}] \cup f[\{n+1\}] = f[\{1, \ldots, n\}] \cup \{f(n+1)\}$.

Let us show that $\bigcap(A \cup B) = (\bigcap A) \cap (\bigcap B)$ for any nonempty systems of sets A, B. Let A, B be nonempty systems of sets. $\bigcap(A \cup B) \subseteq (\bigcap A) \cap (\bigcap B)$. $((\bigcap A) \cap (\bigcap B)) \subseteq \bigcap(A \cup B)$. End.

Hence $\bigcap U = \bigcap(V \cup \{f(n+1)\}) = (\bigcap V) \cap (\bigcap\{f(n+1)\}) = (\bigcap V) \cap f(n+1) \in X$. Indeed V and $\{f(n+1)\}$ are nonempty systems of sets. End. End. Qed.

Therefore Φ contains every natural number. Thus $\bigcap U \in X$ for every nonempty finite subset U of X. Consequently X is closed under finite intersections. End. $\square$