Chapter 1

Ordering

File: `arithmetic/sections/04_ordering.ftl.tex`

[readtex `foundations/sections/11_binary-relations.ftl.tex`]

[readtex `arithmetic/sections/03_addition.ftl.tex`]

1.1 Definitions and immediate consequences

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Definition 1.1. Let n, m be natural numbers. $n < m$ iff there exists a nonzero natural number k such that $m = n + k$.

Let n is less than m stand for $n < m$. Let $n > m$ stand for $m < n$. Let n is greater than m stand for $n > m$. Let $n \nless m$ stand for n is not less than m. Let $n \ngtr m$ stand for n is not greater than m.

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Definition 1.2. Let n be a natural number. $\mathbb{N}_{<n} = \{k \in \mathbb{N} \mid k < n\}$.

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Definition 1.3. Let n be a natural number. $\mathbb{N}_{>n} = \{k \in \mathbb{N} \mid k > n\}$.

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Definition 1.4. Let n be a natural number. n is positive iff $n > 0$.

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Definition 1.5. Let n, m be natural numbers. $n \leq m$ iff there exists a natural number k such that $m = n + k$.

Let n is less than or equal to m stand for $n \leq m$. Let $n \geq m$ stand for $m \leq n$. Let n is greater than or equal to m stand for $n \geq m$. Let $n \nleq m$ stand for n is not less than or equal to m. Let $n \ngeq m$ stand for n is not greater than or equal to m.

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Definition 1.6. Let n be a natural number. $\mathbb{N}_{\leq n} = \{k \in \mathbb{N} \mid k \leq n\}$.

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Definition 1.7. Let n be a natural number. $\mathbb{N}_{\geq n} = \{k \in \mathbb{N} \mid k \geq n\}$.

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Proposition 1.8. Let n, m be natural numbers. $n \leq m$ iff $n < m$ or $n = m$.

Proof. Case $n \leq m$. Take a natural number k such that $m = n + k$. If $k = 0$ then $n = m$. If $k \neq 0$ then $n < m$. End.

Case $n < m$ or $n = m$. If $n < m$ then there is a positive natural number k such that $m = n + k$. If $n = m$ then $m = n + 0$. Thus if $n < m$ then there is a natural number k such that $m = n + k$. End. □

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Definition 1.9. Let n be a natural number. A predecessor of n is a natural number that is less than n.

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Definition 1.10. Let n be a natural number. A successor of n is a natural number that is greater than n.

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Proposition 1.11. Let n be a natural number. Then n is positive iff n is nonzero.

Proof. Case n is positive. Take a positive natural number k such that $n = 0 + k = k$. Then we have $n \neq 0$. End.

Case n is nonzero. Take a natural number k such that $n = k+1$. Then $n = 0+(k+1)$. $k + 1$ is positive. Hence $0 < n$. End. □

1.2 Basic properties

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Proposition 1.12. Let n be a natural number. Then

$$n \not< n.$$

Proof. Assume the contrary. Then we can take a positive natural number k such that $n = n + k$. Then we have $0 = k$. Contradiction. □

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Proposition 1.13. Let n, m be natural numbers. Then

$$n < m \quad \text{implies} \quad n \neq m.$$

Proof. Assume $n < m$. Take a positive natural number k such that $m = n + k$. If $n = m$ then $k = 0$. Hence $n \neq m$. □

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Proposition 1.14. Let n, m be natural numbers. Then

$$(n \leq m \text{ and } m \leq n) \quad \text{implies} \quad n = m.$$

Proof. Assume $n \leq m$ and $m \leq n$. Take natural numbers k, l such that $m = n + k$ and $n = m + l$. Then $m = (m + l) + k = m + (l + k)$. Hence $l + k = 0$. Thus $l = 0 = k$. Indeed if $l \neq 0$ or $k \neq 0$ then $l + k$ is the direct successor of some natural number. Therefore $m = n$. □

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Proposition 1.15. Let n, m, k be natural numbers. Then

$$n < m < k \quad \text{implies} \quad n < k.$$

Proof. Assume $n < m < k$. Take a positive natural number a such that $m = n + a$. Take a positive natural number b such that $k = m + b$. Then $k = (n + a) + b = n + (a + b)$. $a + b$ is positive. Hence $n < k$. □

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Proposition 1.16. Let n, m, k be natural numbers. Then

$$n \leq m \leq k \quad \text{implies} \quad n \leq k.$$

Proof. Assume $n \leq m \leq k$. Case $n = m = k$. Obvious. Case $n = m < k$. Obvious. Case $n < m = k$. Obvious. Case $n < m < k$. Obvious. □

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Proposition 1.17. Let n, m, k be natural numbers. Then

$$n \leq m < k \quad \text{implies} \quad n < k.$$

Proof. Assume $n \leq m < k$. If $n = m$ then $n < k$. If $n < m$ then $n < k$. □

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Proposition 1.18. Let n, m, k be natural numbers. Then

$$n < m \leq k \quad \text{implies} \quad n < k.$$

Proof. Assume $n < m \leq k$. If $m = k$ then $n < k$. If $m < k$ then $n < k$. ☐

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Proposition 1.19. Let n, m be natural numbers. Then

$$n < m \quad \text{implies} \quad n + 1 \leq m.$$

Proof. Assume $n < m$. Take a positive natural number k such that $m = n + k$.

Case $k = 1$. Then $m = n + 1$. Hence $n + 1 \leq m$. End.

Case $k \neq 1$. Then we can take a natural number l such that $k = l + 1$. Then $m = n + (l + 1) = (n + l) + 1 = (n + 1) + l$. l is positive. Thus $n + 1 < m$. End. ☐

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Proposition 1.20. Let n, m be natural numbers. Then $n < m$ or $n = m$ or $n > m$.

Proof. Define $\Phi = \{m' \in \mathbb{N} \mid n < m' \text{ or } n = m' \text{ or } n > m'\}$.

(1) Φ contains 0.

(2) For all $m' \in \Phi$ we have $m' + 1 \in \Phi$.
Proof. Let $m' \in \Phi$.

Case $n < m'$. Obvious.

Case $n = m'$. Obvious.

Case $n > m'$. Take a positive natural number k such that $n = m' + k$.

Case $k = 1$. Obvious.

Case$k \neq 1$. Take a natural number l such that $n = (m' + 1) + l$. Hence $n > m' + 1$. Indeed l is positive. End. Qed. Qed.

Thus every natural number is contained in Φ. Therefore $n < m$ or $n = m$ or $n > m$. ☐

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Proposition 1.21. Let n, m be natural numbers. Then

$$n \not< m \quad \text{iff} \quad n \geq m.$$

Proof. Case $n \not< m$. Then $n = m$ or $n > m$. Hence $n \geq m$. End.

Case $n \geq m$. Assume $n < m$. Then $n \leq m$. Hence $n = m$. Contradiction. End. □

1.3 Ordering and successors

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Proposition 1.22. Let n, m be natural numbers. Then

$$n < m \leq n + 1 \quad \text{implies} \quad m = n + 1.$$

Proof. Assume $n < m \leq n+1$. Take a positive natural number k such that $m = n+k$. Take a natural number l such that $n+1 = m+l$. Then $n+1 = m+l = (n+k)+l = n+(k+l)$. Hence $k+l = 1$.

We have $l = 0$.
Proof. Assume the contrary. Then $k, l > 0$.

Case $k, l = 1$. Then $k + l = 2 \neq 1$. Contradiction. End.

Case $k = 1 and l \neq 1$. Then $l > 1$. Hence $k + l > 1 + l > 1$. Contradiction. End.

Case $k \neq 1 and l = 1$. Then $k > 1$. Hence $k + l > k + 1 > 1$. Contradiction. End.

Case $k, l \neq 1$. Take natural numbers a, b such that $k = a + 1$ and $l = b + 1$. Indeed $k, l \neq 0$. Hence $k = a + 1$ and $l = b + 1$. Thus $k, l > 1$. Indeed a, b are positive. End. Qed.

Then we have $n + 1 = m + l = m + 0 = m$. □

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Proposition 1.23. Let n, m be natural numbers. Then

$$n \leq m < n + 1 \quad \text{implies} \quad n = m.$$

Proof. Assume $n \leq m < n + 1$.

Case $n = m$. Obvious.

Case $n < m$. Then $n < m \leq n + 1$. Hence $n = m$. End. □

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Corollary 1.24. Let n be a natural number. There is no natural number m such that $n < m < n + 1$.

Proof. Assume the contrary. Take a natural number m such that $n < m < n + 1$. Then $n < m \leq n + 1$ and $n \leq m < n + 1$. Hence $m = n + 1$ and $m = n$. Hence $n = n + 1$. Contradiction. □

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Proposition 1.25. Let n be a natural number. Then

$$n + 1 \geq 1.$$

Proof. Case $n = 0$. Obvious.

Case $n \neq 0$. Then $n > 0$. Hence $n + 1 > 0 + 1 = 1$. End. □

1.4 Ordering and addition

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Proposition 1.26. Let n, m, k be natural numbers. Then

$$n < m \quad \text{iff} \quad n + k < m + k.$$

Proof. Case $n < m$. Take a positive natural number l such that $m = n + l$. Then $m + k = (n + l) + k = (n + k) + l$. Hence $n + k < m + k$. End.

Case $n + k < m + k$. Take a positive natural number l such that $m + k = (n + k) + l$. $(n + k) + l = n + (k + l) = n + (l + k) = (n + l) + k$. Hence $m + k = (n + l) + k$. Thus $m = n + l$. Therefore $n < m$. End. □

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Corollary 1.27. Let n, m, k be natural numbers. Then

$$n < m \quad \text{iff} \quad k + n < k + m.$$

Proof. We have $k + n = n + k$ and $k + m = m + k$. Hence $k + n < k + m$ iff $n + k < m + k$. □

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Corollary 1.28. Let n, m, k be natural numbers. Then

$$n \leq m \quad \text{iff} \quad k + n \leq k + m.$$

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Corollary 1.29. Let n, m, k be natural numbers. Then

$$n \leq m \quad \text{iff} \quad n + k \leq m + k.$$

1.5 The natural numbers are well-ordered

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Definition 1.30.

$$< = \{(n, m) \mid n \text{ and } m \text{ are natural numbers such that } n < m\}.$$

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Proposition 1.31. Let A be a nonempty subclass of $\mathbb{N}$. Let n, m be least elements of A regarding $<$. Then $n = m$.

Proof. Assume $n \neq m$. Then $n < m$ or $m < n$. If $n < m$ then $n \notin A$. If $m < n$ then $m \notin A$. Hence $n, m \notin A$. Contradiction. Therefore $n = m$. □

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Proposition 1.32. Let A be a nonempty subclass of $\mathbb{N}$. Then A has a least element regarding $<$.

Proof. Assume the contrary.

Let us show that for each $n \in A$ there exists a $m \in A$ such that $m < n$. Let $n \in A$. A has no least element regarding $<$. Assume that there exists no $m \in A$ such that $m < n$. Then $n \leq m$ for all $m \in A$. Hence n is a least element of A regarding $<$. Contradiction. End.

Define $\Phi = \{n \in \mathbb{N} \mid n \text{ is less than any element of } A\}$.

(1) Φ contains 0.
Proof. $0 \notin A$. Hence 0 is less than every element of A. Thus $0 \in \Phi$. Qed.

(2) For all $n \in \Phi$ we have $n + 1 \in \Phi$.
Proof. Let $n \in \Phi$. Then n is less than any element of A. Assume that Φ does not contain $n+1$. Then we can take an $m \in A$ such that $n+1 \not< m$. Then $n < m \leq n+1$. Hence $m = n + 1$. Thus $n + 1$ is a least element of A regarding $<$. Contradiction. Qed.

Then Φ contains every natural number. Therefore every natural number is less than any element of A. Consequently A is empty. Contradiction. □

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Corollary 1.33. $<$ is a wellorder on every nonempty subclass of $\mathbb{N}$.

Proof. Let A be a nonempty subclass of $\mathbb{N}$. For any $n, m \in A$ we have $(n, m) \in <$ iff $n < m$.

(1) $<$ is irreflexive on A. Indeed for any $n \in A$ we have $n \not< n$.

(2) $<$ is transitive on A. Indeed for any $n, m, k \in A$ if $n < m$ and $m < k$ then $n < k$.

(3) $<$ is connected on A. Indeed for any distinct $n, m \in A$ we have $n < m$ or $m < n$.

Hence $<$ is a strict linear order on A. $<$ is wellfounded on A. Indeed every nonempty subclass of A has a least element regarding $<$. Thus $<$ is a wellorder on A. □

1.6 Induction revisited

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Theorem 1.34. Let A be a class. Assume for all $n \in \mathbb{N}$ if A contains all predecessors of n then A contains n. Then A contains every natural number.

Proof. Assume the contrary. Take a natural number n that is not contained in A. Then n is contained in $\mathbb{N} \setminus A$. Hence we can take a least element m of $\mathbb{N} \setminus A$ regarding $<$. Then $\mathbb{N} \setminus A$ does not contain any predecessor of m. Therefore A contains all predecessors of m. Consequently A contains m. Contradiction. □

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Theorem 1.35. Let A be a class. Let k be a natural number such that $k \in A$. Assume that for all $n \in \mathbb{N}_{\geq k}$ if $n \in A$ then $n + 1 \in A$. Then for all $n \in \mathbb{N}_{\geq k}$ we have $n \in A$.

Proof. Define $\Phi = \{n \in \mathbb{N} \mid \text{if } n \geq k \text{ then } n \in A\}$.

(1) Φ contains 0. Indeed if $0 \geq k$ then $0 = k \in A$.

(2) For all $n \in \Phi$ we have $n + 1 \in \Phi$.
Proof. Let $n \in \Phi$.

Let us show that if $n + 1 \geq k$ then $n + 1 \in A$. Assume $n + 1 \geq k$.

Case $n < k$. Then $n + 1 = k$. Hence $n + 1 \in A$. End.

Case $n \geq k$. Then $n \in A$. Hence $n + 1 \in A$. End. End.

Therefore $n + 1 \in \Phi$. Qed.

Thus Φ contains every natural number. Consequently for all $n \in \mathbb{N}_{\geq k}$ we have $n \in A$. □