Chapter 1

Addition

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[readtex `arithmetic/sections/02_recursion.ftl.tex`]

1.1 Definition of addition

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Lemma 1.1. There exists a $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$ we have $\varphi(n, 0) = n$ and $\varphi(n, \mathrm{succ}(m)) = \mathrm{succ}(\varphi(n, m))$ for all $m \in \mathbb{N}$.

Proof. Take $A = [\mathbb{N} \to \mathbb{N}]$. Define $a(n) = n$ for $n \in \mathbb{N}$. Then A is a set and $a \in A$.

[skipfail on] Define $f(g) = \lambda n \in \mathbb{N}.\, \mathrm{succ}(g(n))$ for $g \in A$. [skipfail off]

Then $f : A \to A$. Indeed $f(g)$ is a map from $\mathbb{N}$ to $\mathbb{N}$ for any $g \in A$. Consider a $\psi : \mathbb{N} \to A$ such that ψ is recursively defined by a and f (by **??**). Define $\varphi(n, m) = \psi(m)(n)$ for $(n, m) \in \mathbb{N} \times \mathbb{N}$. Then φ is a map from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$.

(1) For all $n \in \mathbb{N}$ we have $\varphi(n, 0) = n$.
Proof. Let $n \in \mathbb{N}$. Then $\varphi(n, 0) = \psi(0)(n) = a(n) = n$. Qed.

(2) For all $n, m \in \mathbb{N}$ we have $\varphi(n, \mathrm{succ}(m)) = \mathrm{succ}(\varphi(n, m))$.
Proof. Let $n, m \in \mathbb{N}$. Then $\varphi(n, \mathrm{succ}(m)) = \psi(\mathrm{succ}(m))(n) = f(\psi(m))(n) = \mathrm{succ}(\psi(m)(n)) = \mathrm{succ}(\varphi(n, m))$. Qed. □

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Lemma 1.2. Let $\varphi, \varphi' : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Assume that for all $n \in \mathbb{N}$ we have $\varphi(n, 0) = n$ and $\varphi(n, \mathrm{succ}(m)) = \mathrm{succ}(\varphi(n, m))$ for all $m \in \mathbb{N}$. Assume that for all $n \in \mathbb{N}$ we have $\varphi'(n, 0) = n$ and $\varphi'(n, \mathrm{succ}(m)) = \mathrm{succ}(\varphi'(n, m))$ for all $m \in \mathbb{N}$. Then $\varphi = \varphi'$.

Proof. Define $\Phi = \{m \in \mathbb{N} \mid \varphi(n, m) = \varphi'(n, m) \text{ for all } n \in \mathbb{N}\}$.

(1) $0 \in \Phi$. Indeed $\varphi(n, 0) = n = \varphi'(n, 0)$ for all $n \in \mathbb{N}$.

(2) For all $m \in \Phi$ we have $\mathrm{succ}(m) \in \Phi$.
Proof. Let $m \in \Phi$. Then $\varphi(n, m) = \varphi'(n, m)$ for all $n \in \mathbb{N}$. Hence $\varphi(n, \mathrm{succ}(m)) = \mathrm{succ}(\varphi(n, m)) = \mathrm{succ}(\varphi'(n, m)) = \varphi(n, \mathrm{succ}(m))$ for all $n \in \mathbb{N}$. Qed.

Thus Φ contains every natural number. Therefore $\varphi(n, m) = \varphi'(n, m)$ for all $n, m \in \mathbb{N}$. □

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Definition 1.3. add is the map from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ such that for all $n \in \mathbb{N}$ we have $\mathrm{add}(n, 0) = n$ and $\mathrm{add}(n, \mathrm{succ}(m)) = \mathrm{succ}(\mathrm{add}(n, m))$ for all $m \in \mathbb{N}$.

Let $n + m$ stand for $\mathrm{add}(n, m)$. Let the sum of n and m stand for $n + m$.

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Lemma 1.4. Let n, m be natural numbers. Then $(n, m) \in \mathrm{dom}(\mathrm{add})$.

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Lemma 1.5. Let n, m be natural numbers. Then $n + m$ is a natural number.

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Lemma 1.6. Let n be a natural number. Then $\mathrm{succ}(n) = n + 1$.

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Lemma 1.7. Let n be a natural number. Then $n + 0 = n$.

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Lemma 1.8. Let n, m be natural numbers. Then $n + (m + 1) = (n + m) + 1$.

1.2 The Peano axioms and recursion, revisited

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Proposition 1.9. Let n, m be natural numbers. If $n + 1 = m + 1$ then $n = m$.

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Proposition 1.10. Let n be a natural number. Then $n + 1 \neq 0$.

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Proposition 1.11 (Induction). Let A be a class. Assume $0 \in A$. Assume that for all $n \in \mathbb{N}$ if $n \in A$ then $n + 1 \in A$. Then A contains every natural number.

Proposition 1.12. Let a be an object and f be a map. Let φ be a map from $\mathbb{N}$ to $\mathrm{dom}(f)$. φ is recursively defined by a and f iff $\varphi(0) = a$ and $\varphi(n+1) = f(\varphi(n))$ for every $n \in \mathbb{N}$.

1.3 Computation laws

Associativity

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Proposition 1.13. Let n, m, k be natural numbers. Then

$$n + (m + k) = (n + m) + k.$$

Proof. Define $\Phi = \{k' \in \mathbb{N} \mid n + (m + k') = (n + m) + k'\}$.

(1) 0 is contained in Φ. Indeed $n + (m + 0) = n + m = (n + m) + 0$.

(2) For all $k' \in \Phi$ we have $k' + 1 \in \Phi$.
Proof. Let $k' \in \Phi$. Then $n + (m + k') = (n + m) + k'$. Hence

$$\begin{aligned} & n + (m + (k' + 1)) \\ &= n + ((m + k') + 1) \\ &= (n + (m + k')) + 1 \\ &= ((n + m) + k') + 1 \\ &= (n + m) + (k' + 1). \end{aligned}$$

Thus $k' + 1 \in \Phi$. Qed.

Thus every natural number is an element of Φ. Therefore $n + (m + k) = (n + m) + k$. □

Commutativity

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Proposition 1.14. Let n, m be natural numbers. Then

$$n + m = m + n.$$

Proof. Define $\Phi = \{m' \in \mathbb{N} \mid n + m' = m' + n\}$.

(1) 0 is an element of Φ.
Proof. Define $\Psi = \{n' \in \mathbb{N} \mid n' + 0 = 0 + n'\}$.

(1a) 0 belongs to Ψ.

(1b) For all $n' \in \Psi$ we have $n' + 1 \in \Psi$.

Proof. Let $n' \in \Psi$. Then $n' + 0 = 0 + n'$. Hence

$$(n' + 1) + 0$$

$$= n' + 1$$

$$= (n' + 0) + 1$$

$$= (0 + n') + 1$$

$$= 0 + (n' + 1).$$

Qed.

Hence every natural number belongs to Ψ. Thus $n + 0 = 0 + n$. Therefore 0 is an element of Φ. Qed.

Let us show that (2) $n + 1 = 1 + n$.
Proof. Define $\Theta = \{n' \in \mathbb{N} \mid n' + 1 = 1 + n'\}$.

(2a) 0 is an element of Θ.

(2b) For all $n' \in \Theta$ we have $n' + 1 \in \Theta$.
Proof. Let $n' \in \Theta$. Then $n' + 1 = 1 + n'$. Hence

$$(n' + 1) + 1$$

$$= (1 + n') + 1$$

$$= 1 + (n' + 1).$$

Thus $n' + 1 \in \Theta$. Qed.

Thus every natural number belongs to Θ. Therefore $n + 1 = 1 + n$. Qed.

(3) For all $m' \in \Phi$ we have $m' + 1 \in \Phi$.
Proof. Let $m' \in \Phi$. Then $n + m' = m' + n$. Hence

$$n + (m' + 1)$$

$$= (n + m') + 1$$

$$= (m' + n) + 1$$

$$= m' + (n + 1)$$

$$= m' + (1 + n)$$

$$= (m' + 1) + n.$$

Thus $m' + 1 \in \Phi$. Qed.

Thus every natural number is an element of Φ. Therefore $n + m = m + n$. □

Cancellation

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Proposition 1.15. Let n, m, k be natural numbers. Then

$$n + k = m + k \quad \text{implies} \quad n = m.$$

Proof. Define $\Phi = \{k' \in \mathbb{N} \mid \text{if } n + k' = m + k' \text{ then } n = m\}$.

(1) 0 is an element of Φ.

(2) For all $k' \in \Phi$ we have $k' + 1 \in \Phi$.
Proof. Let $k' \in \Phi$. Suppose $n+(k'+1) = m+(k'+1)$. Then $(n+k')+1 = (m+k')+1$. Hence $n + k' = m + k'$. Thus $n = m$. Qed.

Therefore every natural number is an element of Φ. Consequently if $n + k = m + k$ then $n = m$. ☐

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Corollary 1.16. Let n, m, k be natural numbers. Then

$$k + n = k + m \quad \text{implies} \quad n = m.$$

Proof. Assume $k + n = k + m$. We have $k + n = n + k$ and $k + m = m + k$. Hence $n + k = m + k$. Thus $n = m$. ☐

Zero sums

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Proposition 1.17. Let n, m be natural numbers. If $n + m = 0$ then $n = 0$ and $m = 0$.

Proof. Assume $n + m = 0$. Suppose $n \neq 0$ or $m \neq 0$. Then we can take a $k \in \mathbb{N}$ such that $n = k + 1$ or $m = k + 1$. Hence there exists a natural number l such that $n + m = l + (k + 1) = (l + k) + 1 \neq 0$. Contradiction. ☐