Chapter 1

Recursion

File: arithmetic/sections/02_recursion.ftl.tex

[readtex arithmetic/sections/01_natural-numbers.ftl.tex]

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Definition 1.1. Let a be an object and f be a map. Let φ be a map from $\mathbb{N}$ to $\mathrm{dom}(f)$. φ is recursively defined by a and f iff $\varphi(0) = a$ and $\varphi(\mathrm{succ}(n)) = f(\varphi(n))$ for every $n \in \mathbb{N}$.

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Theorem 1.2 (Dedekind). Let A be a set and $a \in A$ and $f : A \to A$. Then there exists a $\varphi : \mathbb{N} \to A$ that is recursively defined by a and f.

Proof. Define

$$\Phi = \left\{ H \in \mathcal{P}(\mathbb{N} \times A) \;\middle|\; \begin{array}{l} (0, a) \in H \text{ and for all } n \in \mathbb{N} \text{ and all } x \in A \text{ if } (n, x) \in H \text{ then} \\ (\mathrm{succ}(n), f(x)) \in H \end{array} \right\}.$$

Let us show that $\bigcap \Phi \in \Phi$.
Proof.

(1) $\bigcap \Phi \in \mathcal{P}(\mathbb{N} \times A)$.
Proof. We have $\mathbb{N} \times A \in \Phi$. Hence Φ is nonempty. Any element of $\bigcap \Phi$ is contained

in every element of Φ. Hence any element of $\bigcap \Phi$ is contained in $\mathbb{N} \times A$. Thus $\bigcap \Phi \subseteq \mathbb{N} \times A$. $\bigcap \Phi$ is a set. Hence $\bigcap \Phi$ is a subset of $\mathbb{N} \times A$. Qed.

(2) $(0, a) \in \bigcap \Phi$.
Indeed $(0, a) \in \mathbb{N} \times A \in \Phi$.

(3) For all $n \in \mathbb{N}$ and all $x \in A$ if $(n, x) \in \bigcap \Phi$ then $(\mathrm{succ}(n), f(x)) \in \bigcap \Phi$.
Proof. Let $n \in \mathbb{N}$ and $x \in A$. Assume $(n, x) \in \bigcap \Phi$. Then (n, x) is contained in every element of Φ. Hence $(\mathrm{succ}(n), f(x))$ is contained in every element of Φ. Thus $(\mathrm{succ}(n), f(x)) \in \bigcap \Phi$. Qed. Qed.

Let us show that for any $n \in \mathbb{N}$ there exists an $x \in A$ such that $(n, x) \in \bigcap \Phi$.
Proof. Define $\Psi = \{n \in \mathbb{N} \mid \text{there exists an } x \in A \text{ such that } (n, x) \in \bigcap \Phi\}$.

(1) 0 is contained in Ψ. Indeed $(0, a) \in \bigcap \Phi$.

(2) For all $n \in \Psi$ we have $\mathrm{succ}(n) \in \Psi$.
Proof. Let $n \in \Psi$. Take an $x \in A$ such that $(n, x) \in \bigcap \Phi$. Then $(\mathrm{succ}(n), f(x)) \in \bigcap \Phi$. Hence $\mathrm{succ}(n) \in \Psi$. Indeed $f(x) \in A$. Qed. Qed.

Let us show that for all $n \in \mathbb{N}$ and all $x, y \in A$ if $(n, x), (n, y) \in \bigcap \Phi$ then $x = y$.
Proof. (a) Define $\Theta = \{n \in \mathbb{N} \mid \text{for all } x, y \in A \text{ if } (n, x), (n, y) \in \bigcap \Phi \text{ then } x = y\}$.

(1) Θ contains 0.
Proof. Let us show that for all $x, y \in A$ if $(0, x), (0, y) \in \bigcap \Phi$ then $x = y$. Let $x, y \in A$. Assume $(0, x), (0, y) \in \bigcap \Phi$.

Let us show that $x, y = a$. Assume $x \neq a$ or $y \neq a$.

Case $x \neq a$. $(0, x), (0, a)$ are contained in every element of Φ. Then $(0, x), (0, a) \in \bigcap \Phi$. Take $H = (\bigcap \Phi) \setminus \{(0, x)\}$.

Let us show that $H \in \Phi$. (1) $H \in \mathcal{P}(\mathbb{N} \times A)$.

(2) $(0, a) \in H$.

(3) For all $n \in \mathbb{N}$ and all $b \in A$ if $(n, b) \in H$ then $(\mathrm{succ}(n), f(b)) \in H$.
Proof. Let $n \in \mathbb{N}$ and $b \in A$. Assume $(n, b) \in H$. Then $(\mathrm{succ}(n), f(b)) \in \bigcap \Phi$. We have $(\mathrm{succ}(n), f(b)) \neq (0, x)$. Hence $(\mathrm{succ}(n), f(b)) \in H$. Qed. End.

Then $(0, x)$ is not contained in every member of Φ. Contradiction. End.

Case $y \neq a$. $(0, y), (0, a)$ are contained in every element of Φ. Then $(0, y), (0, a) \in \bigcap \Phi$. Take $H = (\bigcap \Phi) \setminus \{(0, y)\}$.

Let us show that $H \in \Phi$. (1) $H \in \mathcal{P}(\mathbb{N} \times A)$.

(2) $(0, a) \in H$.

(3) For all $n \in \mathbb{N}$ and all $b \in A$ if $(n, b) \in H$ then $(\mathrm{succ}(n), f(b)) \in H$.
Proof. Let $n \in \mathbb{N}$ and $b \in A$. Assume $(n, b) \in H$. Then $(\mathrm{succ}(n), f(b)) \in \bigcap \Phi$. We have $(\mathrm{succ}(n), f(b)) \neq (0, y)$. Hence $(\mathrm{succ}(n), f(b)) \in H$. Qed. End.

Then $(0, y)$ is not contained in every member of Φ. Contradiction. End. End. End.

Qed.

(2) For all $n \in \Theta$ we have $\mathrm{succ}(n) \in \Theta$.
Proof. Let $n \in \Theta$. Then for all $x, y \in A$ if $(n, x), (n, y) \in \bigcap \Phi$ then $x = y$. Consider a $b \in A$ such that $(n, b) \in \bigcap \Phi$. Then $(\mathrm{succ}(n), f(b)) \in \bigcap \Phi$.

Let us show that for all $x, y \in A$ if $(\mathrm{succ}(n), x), (\mathrm{succ}(n), y) \in \bigcap \Phi$ then $x = f(b) = y$. Let $x, y \in A$. Assume $(\mathrm{succ}(n), x), (\mathrm{succ}(n), y) \in \bigcap \Phi$. Suppose $x \neq f(b)$ or $y \neq f(b)$.

Case $x \neq f(b)$. Take $H = (\bigcap \Phi) \setminus \{(\mathrm{succ}(n),\, x)\}$.

(a) $H \in \mathcal{P}(\mathbb{N} \times A)$.

(b) $(0, a) \in H$. Indeed $(0, a) \in \bigcap \Phi$ and $(0, a) \notin \{(\mathrm{succ}(n),\, x)\}$.

(c) For all $m \in \mathbb{N}$ and all $z \in A$ if $(m, z) \in H$ then $(\mathrm{succ}(m), f(z)) \in H$.
Proof. Let $m \in \mathbb{N}$ and $z \in A$. Assume $(m, z) \in H$. Then $(m, z) \in \bigcap \Phi$. Hence $(\mathrm{succ}(m), f(z)) \in \bigcap \Phi$. $(\mathrm{succ}(m), f(z)) \neq (\mathrm{succ}(n), x)$. Therefore $(\mathrm{succ}(m), f(z)) \in H$. Qed.

Thus $H \in \Phi$. Therefore every element of $\bigcap \Phi$ is contained in H. Consequently $(\mathrm{succ}(n), x) \in H$. Contradiction. End.

Case $y \neq f(b)$. Take a class H such that $H = (\bigcap \Phi) \setminus \{(\mathrm{succ}(n),\, y)\}$.

(a) $H \in \mathcal{P}(\mathbb{N} \times A)$.

(b) $(0, a) \in H$. Indeed $(0, a) \in \bigcap \Phi$ and $(0, a) \notin \{(\mathrm{succ}(n),\, y)\}$.

(c) For all $m \in \mathbb{N}$ and all $z \in A$ if $(m, z) \in H$ then $(\mathrm{succ}(m), f(z)) \in H$.
Proof. Let $m \in \mathbb{N}$ and $z \in A$. Assume $(m, z) \in H$. Then $(m, z) \in \bigcap \Phi$. Hence $(\mathrm{succ}(m), f(z)) \in \bigcap \Phi$. $(\mathrm{succ}(m), f(z)) \neq (\mathrm{succ}(n), y)$. Therefore $(\mathrm{succ}(m), f(z)) \in H$. Qed.

Thus $H \in \Phi$. Therefore every element of $\bigcap \Phi$ is contained in H. Consequently $(\mathrm{succ}(n), y) \in H$. Contradiction. End.

Hence it is wrong that $x \neq f(b)$ or $y \neq f(b)$. Consequently $x = f(b) = y$. End.

Therefore $\mathrm{succ}(n) \in \Theta$ (by a). Qed. Qed.

Define $\varphi(n) =$ "choose $x \in A$ such that $(n, x) \in \bigcap \Phi$ in x" for $n \in \mathbb{N}$.

(1) Then φ is a map from $\mathbb{N}$ to A and we have $\varphi(0) = a$.

(2) For all $n \in \mathbb{N}$ we have $\varphi(\mathrm{succ}(n)) = f(\varphi(n))$.
Proof. Let $n \in \mathbb{N}$. Take $x \in A$ such that $\varphi(n) = x$. Then $(n, x) \in \bigcap \Phi$. Hence $(\mathrm{succ}(n), f(\varphi(n))) = (\mathrm{succ}(n), f(x)) \in \bigcap \Phi$. Thus $\varphi(\mathrm{succ}(n)) = f(\varphi(n))$. Qed. □

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Proposition 1.3. Let A be a set and $a \in A$ and $f : A \to A$. Let $\varphi, \varphi' : \mathbb{N} \to A$. Assume that φ and φ' are recursively defined by a and f. Then $\varphi = \varphi'$.

Proof. Define $\Phi = \{n \in \mathbb{N} \mid \varphi(n) = \varphi'(n)\}$.

(1) Φ contains 0. Indeed $\varphi(0) = a = \varphi'(0)$.

(2) For all $n \in \Phi$ we have $\mathrm{succ}(n) \in \Phi$.
Proof. Let $n \in \Phi$. Then $\varphi(n) = \varphi'(n)$. Hence $\varphi(\mathrm{succ}(n)) = f(\varphi(n)) = f(\varphi'(n)) = \varphi'(\mathrm{succ}(n))$. Qed.

Thus Φ contains every natural number. Consequently $\varphi(n) = \varphi'(n)$ for each $n \in \mathbb{N}$. □