Chapter 1

Natural numbers

File: arithmetic/sections/01_natural-numbers.ftl.tex

[readtex foundations/sections/10_sets.ftl.tex]

1.1 The language of Peano Arithmetic

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Signature 1.1. A natural number is an object.

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Definition 1.2. $\mathbb{N}$ is the class of natural numbers.

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Signature 1.3. 0 is a natural number.

Let zero stand for 0. Let n is nonzero stand for $n \neq 0$.

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Signature 1.4. Let n be a natural number. $\mathrm{succ}(n)$ is a natural number.

Let the direct successor of n stand for $\mathrm{succ}(n)$.

1.2 The Peano Axioms

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Axiom 1.5. Let n, m be natural numbers. If $\mathrm{succ}(n) = \mathrm{succ}(m)$ then $n = m$.

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Axiom 1.6. There exists no natural number n such that $\mathrm{succ}(n) = 0$.

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Axiom 1.7. Let Φ be a class. Assume $0 \in \Phi$ and for all natural numbers n if $n \in \Phi$ then $\mathrm{succ}(n) \in \Phi$. Then Φ contains every natural number.

1.3 Immediate consequences

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Proposition 1.8. Let n be a natural number. Then $n = 0$ or $n = \mathrm{succ}(m)$ for some natural number m.

Proof. Define $\Phi = \{n' \in \mathbb{N} \mid n' = 0 \text{ or } n' = \mathrm{succ}(m') \text{ for some natural number } m'\}$. $0 \in \Phi$ and for all $n' \in \Phi$ we have $\mathrm{succ}(n') \in \Phi$. Hence every natural number is contained in Φ. Thus $n = 0$ or $n = \mathrm{succ}(m)$ for some natural number m. □

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Proposition 1.9. Let n be a natural number. Then $n \neq \mathrm{succ}(n)$.

Proof. Define $\Phi = \{n' \in \mathbb{N} \mid n' \neq \mathrm{succ}(n')\}$.

(1) 0 belongs to Φ.

(2) For all $n' \in \Phi$ we have $\mathrm{succ}(n') \in \Phi$.
Proof. Let $n' \in \Phi$. Then $n' \neq \mathrm{succ}(n')$. If $\mathrm{succ}(n') = \mathrm{succ}(\mathrm{succ}(n'))$ then $n' = \mathrm{succ}(n')$. Thus it is wrong that $\mathrm{succ}(n') = \mathrm{succ}(\mathrm{succ}(n'))$. Hence $\mathrm{succ}(n') \in \Phi$. Qed.

Therefore every natural number is an element of Φ. Consequently $n \neq \mathrm{succ}(n)$. □

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Proposition 1.10. $\mathbb{N}$ is a set.

Proof. Define $f(n) = \mathrm{succ}(n)$ for $n \in \mathbb{N}$. Then f is a map from $\mathbb{N}$ to $\mathbb{N}$. Hence we can take a subset X of $\mathbb{N}$ that is inductive regarding 0 and f. Then $0 \in X$ and for all $n \in X$ we have $\mathrm{succ}(n) \in X$. Hence X contains every natural number. Thus we have $\mathbb{N} \subseteq X$ and $X \subseteq \mathbb{N}$. Therefore $\mathbb{N} = X$. Consequently $\mathbb{N}$ is a set. □

1.4 Additional constants

ARITHMETIC_01_7540560137027584

Definition 1.11. $1 = \mathrm{succ}(0)$.

Let one stand for 1.

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Definition 1.12. $2 = \mathrm{succ}(1)$.

Let two stand for 2.

ARITHMETIC_01_3836725109456896

Definition 1.13. $3 = \mathrm{succ}(2)$.

Let three stand for 3.

ARITHMETIC_01_1709884968009728

Definition 1.14. $4 = \text{succ}(3)$.

Let four stand for 4.

ARITHMETIC_01_6734726333202432

Definition 1.15. $5 = \text{succ}(4)$.

Let five stand for 5.

ARITHMETIC_01_949139189792768

Definition 1.16. $6 = \text{succ}(5)$.

Let six stand for 6.

ARITHMETIC_01_7245471749767168

Definition 1.17. $7 = \text{succ}(6)$.

Let seven stand for 7.

ARITHMETIC_01_5658172888973312

Definition 1.18. $8 = \text{succ}(7)$.

Let eight stand for 8.

ARITHMETIC_01_7371844250238976

Definition 1.19. $9 = \text{succ}(8)$.

Let nine stand for 9.